

NUCLEAR SCISSORS MODE WITH PAIRING

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Abstract

The coupled dynamics of the scissors mode and the isovector giant quadrupole resonance are studied using a generalized Wigner function moments method taking into account pair correlations. Equations of motion for angular momentum, quadrupole moment and other relevant collective variables are derived on the basis of the time dependent Hartree-Fock-Bogoliubov equations. Analytical expressions for energy centroids and transitions probabilities are found for the harmonic oscillator model with the quadrupole-quadrupole residual interaction and monopole pairing force. Deformation dependences of energies and $B(M1)$ values are correctly reproduced. The inclusion of pair correlations leads to a drastic improvement in the description of qualitative and quantitative characteristics of the scissors mode.

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1 Introduction

An exhaustive analysis of the coupled dynamics of the scissors mode and the isovector giant quadrupole resonance in a model of harmonic oscillator with quadrupole–quadrupole residual interaction has been performed in [1]. The Wigner Function Moments (WFM) method was applied to derive the dynamical equations for angular momentum and quadrupole moment. Analytical expressions for energies, $B(M1)$ - and $B(E2)$ -values, sum rules and flow patterns of both modes were found for arbitrary values of the deformation parameter. The subtle nature of the phenomenon and its peculiarities were clarified.

Nevertheless, this description was not complete, because pairing was not taken into account. It is well known [2], that pairing is very important for the correct quantitative description of the scissors mode. Moreover, its role is crucial for an explanation of the empirically observed deformation dependence of E_{sc} and $B(M1)_{sc}$.

The prediction of the scissors mode was inspired by the geometrical picture of a counterrotating oscillation of the deformed proton density against the deformed neutron density [3, 4]. Thus, as it is seen from its physical nature, the scissors mode can be observed only in deformed nuclei. Therefore, quite naturally, the question of the deformation dependence of its properties (for example, energy E_{sc} and $B(M1)_{sc}$ value) arises. However, during the first years after its discovery in ^{156}Gd [5] “nearly all experimental data were limited to nuclei of about the same deformation ($\delta \approx 0.20 - 0.25$), and the important aspect of orbital $M1$ strength dependence on δ has not yet been examined”, see ref. [6].

The first investigations of the δ -dependence of E_{sc} and $B(M1)_{sc}$ were performed by W. Ziegler et al. [6], who have studied the chain of isotopes $^{148,150,152,154}\text{Sm}$, and by H. H. Pitz et al. [7] and J. Margraf et al. [8], who have studied the chain of isotopes $^{142,146,148,150}\text{Nd}$. They found that the low-energy $B(M1)$ strength exhibits approximately a quadratic dependence on the deformation δ .

Shortly afterwards it was discovered [9, 10], that in even-even nuclei the total low-energy magnetic dipole strength is closely related to the collective $E2$ strength of the 2_1^+ state and, thus, depends quadratically on the nuclear deformation parameter.

Later J. Enders et al. [11] made a theoretical analysis of experimental data on the scissors

mode in nuclei with $140 < A < 200$. Investigating the sum rules S_{+1} and S_{-1} derived by E. Lipparini and S. Stringari [12, 13], they found that the ratio $\bar{\omega} = S_{+1}/S_{-1}$ is proportional to E_{sc} with very good accuracy: $E_{sc} = 0.44\bar{\omega}$. They also observed that the moment of inertia J_{gsb} of the ground state rotational band and the moment of inertia for the irrotational flow $J_{liq} = \delta^2 J_{rig}$ (where J_{rig} is the rigid body moment of inertia) differ by nearly a constant factor ($K \approx 10$) over the entire region. Using this fact and identifying the gyromagnetic ratio and the moment of inertia of the scissors mode with those of the ground state rotational band, they found with the help of the S_{-1} sum rule, that $B(M1)_{sc}$ is proportional to δ^2 .

So, all the rather numerous experimental data demonstrate undoubtedly the δ^2 dependence of $B(M1)_{sc}$ and the very weak deformation dependence of E_{sc} . On the other hand, at the beginning of the nuclear scissors studies all theoretical models, starting from the first work by Suzuki and Rowe [14], predicted a linear δ -dependence for both, $B(M1)_{sc}$ and E_{sc} .

It turned out that the correct δ -dependence is supplied by the pairing correlations. The effects of the pairing interaction in the description of the scissors mode were evaluated for the first time by Bes and Broglia [15]. They assumed “for simplicity that only the two subsets of levels which are closest to the Fermi level (n_{\perp} and $n_{\perp} + 1$) are affected by the pairing interactions”. In this case $B(M1)$ should be multiplied by the factor $(u_{n_{\perp}}v_{n_{\perp}+1} - u_{n_{\perp}+1}v_{n_{\perp}})^2 \approx (u_{n_{\perp}}^2 - v_{n_{\perp}}^2)^2 = (e_{n_{\perp}}/E_{n_{\perp}})^2$ with $E_i = \sqrt{e_i^2 + \Delta^2}$ and $e_i = \epsilon_i - \mu$, where ϵ_i is a single particle energy, Δ is the gap and μ is the chemical potential. The value of $(e_{n_{\perp}}/E_{n_{\perp}})$ was found by “making use of the fact that the moment of inertia is approximately 1/2 of the rigid body value obtained in the absence of pairing”. Thus, in accordance with the Inglis formula one has

$$1/2 = J/J_{rig} = [(u_{n_{\perp}}^2 - v_{n_{\perp}}^2)^2/2E_{n_{\perp}}]/[1/2e_{n_{\perp}}] = (e_{n_{\perp}}/E_{n_{\perp}})^3 \quad (1)$$

and $e_{n_{\perp}}/E_{n_{\perp}}=0.79$. As a result $B(M1)$ is reduced by the factor $(e_{n_{\perp}}/E_{n_{\perp}})^2 = 0.62$, i.e. the influence of pairing is quite remarkable.

It was, however, noted by Hamamoto and Magnusson [16] that this result holds only for well deformed nuclei, where the equality (1) is valid. In general it is necessary to take into account the δ -dependence of the $e_{n_{\perp}}/E_{n_{\perp}}$ - factor. This was done for the first time in ref. [16]. The authors applied “the method of averaging the position of the chemical potential between the occupied subshell (N, n_{\perp}) and the empty shell $(N, n_{\perp} + 1)$ ” to find that the δ -dependence

of $B(M1)$ is determined by the function

$$Y = (\delta A^{4/3}) \left(\sqrt{1 + 4x^2} + \frac{1}{2x} \ln |\sqrt{1 + 4x^2} + 2x| \right) \frac{1}{x} \left(1 - \frac{1}{x\sqrt{1 + x^2}} \ln |\sqrt{1 + x^2} + x| \right) \quad (2)$$

with $x = (\hbar\omega_0\delta)/(2\Delta)$. In the small deformation limit this function is proportional to δ^2 , while for large δ it deviates remarkably from such a simple dependence. The authors performed also a more realistic QRPA calculation for the Woods-Saxon potential with QQ and $\sigma\sigma$ residual interactions, which confirmed their simplified analytical estimate.

In [17], N. Pietralla et al. established the δ -dependence of the e_{n_\perp}/E_{n_\perp} - factor phenomenologically. They were first to perform the theoretical analysis of the experimental data of the scissors mode in nuclei in the mass region $130 < A < 200$. Following the idea of Bes and Broglia they parametrized the e_{n_\perp}/E_{n_\perp} - factor as

$$E_{n_\perp}/e_{n_\perp} = \sqrt{1 + (b\delta)^2/(a\delta)}.$$

The free parameters a and b were fixed by a fit to the experimental moments of inertia with the help of a formula equivalent to (1)

$$(e_{n_\perp}/E_{n_\perp})^3 = J_{exp}/J_{rig}, \quad (3)$$

where $J_{exp} = 3\hbar^2/E(2_1^+)$ is the effective moment of inertia of the ground state band. In this way it was found that “the centers of gravity of the observed $M1$ strength distributions are always close to 3 MeV”, i.e. “the data exhibit a weak dependence of the scissors mode on the deformation parameter”. They also derived a semiempirical formula for the total $M1$ strength of the scissors mode

$$B(M1; 0_1^+ \rightarrow 1_{sc}^+) = 2.6c_g^2 \frac{\delta^3}{1 + (3\delta)^2} \frac{Z^2}{A^{2/3}} \mu_N^2 \quad (4)$$

($c_g = 0.8$ is the scaling factor of the giromagnetic ratio), which describes very well the experimental data and gives a deformation dependence “practically indistinguishable from the δ^2 dependence”.

A direct way to demonstrate the δ -dependence of the e_{n_\perp}/E_{n_\perp} - factor was suggested in [18]. E. Garrido et al. have shown that it is possible to extract analytically the δ^2 factor from the occupation coefficient $\Phi_{\alpha\beta} = (u_\alpha v_\beta - u_\beta v_\alpha)^2$. Using the definitions of the u_α, v_α coefficients,

it is easy to write $\Phi_{\alpha\beta}$ as

$$\Phi_{\alpha\beta} = \frac{(e_\alpha - e_\beta)^2}{4E_\alpha E_\beta} (1 - P_{\alpha\beta}) \quad (5)$$

with

$$P_{\alpha\beta} = \frac{(E_\alpha - E_\beta)^2}{(e_\alpha - e_\beta)^2} = \frac{1}{2E^2} [z - (z^2 - 4e_{\alpha\beta}^2 E^2)^{1/2}] = \frac{e_{\alpha\beta}^2}{z} [1 + \frac{e_{\alpha\beta}^2 E^2}{z^2} + \dots] \quad (6)$$

and $E = \frac{1}{2}(e_\alpha - e_\beta)$, $e_{\alpha\beta} = \frac{1}{2}(e_\alpha + e_\beta)$, $z = e_{\alpha\beta}^2 + E^2 + \Delta^2 = \frac{1}{2}(e_\alpha^2 + e_\beta^2) + \Delta^2 = \frac{1}{2}(E_\alpha^2 + E_\beta^2)$. For the scissors mode $e_\alpha - e_\beta \simeq \hbar\omega\delta$. The function $(1 - P_{\alpha\beta})/(E_\alpha E_\beta)$ has a regular dependence on δ (no poles), so the coefficient $\Phi_{\alpha\beta}$ and, respectively, $B(M1)_{sc}$ are obviously proportional to δ^2 . For the IVGQR $e_\alpha - e_\beta \simeq 2\hbar\omega$, but its $B(M1)$ is proportional to δ^2 even without pairing due to other reasons (see section 3).

In this paper we generalize the WFM method to take into account pair correlations. This allows us to obtain the correct δ -dependence for E_{sc} and $B(M1)_{sc}$ in a slightly different way than in the papers, cited above.

The paper is organized as follows. In section 2 the moments of Time Dependent Hartree-Fock-Bogoliubov (TDHFB) dynamical equations for normal and abnormal densities are calculated and adequate approximations are introduced to obtain the final set of six dynamical equations for the collective variables. In section 3 these equations are decoupled in the isoscalar and isovector sets and the isovector excitation energies and transitions probabilities are calculated in the framework of the HO+QQ model. The results of calculations are discussed in section 4. Concluding remarks are contained in section 5. Some mathematical details are given in Appendices.

2 Phase space moments of TDHFB equations

The time dependent HFB equations in matrix formulation are [19, 20]

$$i\hbar\dot{\mathcal{R}} = [\mathcal{H}, \mathcal{R}] \quad (7)$$

with

$$\mathcal{R} = \begin{pmatrix} \hat{\rho} & -\hat{\kappa} \\ -\hat{\kappa}^\dagger & 1 - \hat{\rho}^* \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \hat{h} & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\hat{h}^* \end{pmatrix}$$

The normal density matrix $\hat{\rho}$ and Hamiltonian \hat{h} are hermitian; the abnormal density $\hat{\kappa}$ and the pairing gap $\hat{\Delta}$ are skew symmetric

$$\hat{\kappa}^\dagger = -\hat{\kappa}^*, \quad \hat{\Delta}^\dagger = -\hat{\Delta}^*.$$

The detailed form of the HFB equations is

$$\begin{aligned} i\hbar\dot{\hat{\rho}} &= \hat{h}\hat{\rho} - \hat{\rho}\hat{h} - \hat{\Delta}\hat{\kappa}^\dagger + \hat{\kappa}\hat{\Delta}^\dagger, & -i\hbar\dot{\hat{\kappa}} &= -\hat{h}\hat{\kappa} - \hat{\kappa}\hat{h}^* + \hat{\Delta} - \hat{\Delta}\hat{\rho}^* - \hat{\rho}\hat{\Delta}, \\ -i\hbar\dot{\hat{\rho}}^* &= \hat{h}^*\hat{\rho}^* - \hat{\rho}^*\hat{h}^* - \hat{\Delta}^\dagger\hat{\kappa} + \hat{\kappa}^\dagger\hat{\Delta}, & -i\hbar\dot{\hat{\kappa}}^\dagger &= \hat{h}^*\hat{\kappa}^\dagger + \hat{\kappa}^\dagger\hat{h} - \hat{\Delta}^\dagger + \hat{\Delta}^\dagger\hat{\rho} + \hat{\rho}^*\hat{\Delta}^\dagger. \end{aligned} \quad (8)$$

We will work with the Wigner transformation [20] of these equations. The relevant mathematical details can be found in Appendix A. To make the formulae more transparent, in the following we will not specify the spin and isospin indices. The isospin indices will be re-introduced at the end. As a rule, we also will not write out the coordinate dependence (\mathbf{r}, \mathbf{p}) of all functions.

The Wigner transform of (8) can be written as

$$\begin{aligned} i\hbar\dot{f} &= i\hbar\{h, f\} - \Delta\kappa^* + \kappa\Delta^* - \frac{i\hbar}{2}\{\Delta, \kappa^*\} + \frac{i\hbar}{2}\{\kappa, \Delta^*\} \\ &\quad - \frac{\hbar^2}{8}[\{\{\kappa, \Delta^*\}\} - \{\{\Delta, \kappa^*\}\}] + \dots, \\ -i\hbar\dot{\bar{f}} &= i\hbar\{\bar{h}, \bar{f}\} - \Delta^*\kappa + \kappa^*\Delta - \frac{i\hbar}{2}\{\Delta^*, \kappa\} + \frac{i\hbar}{2}\{\kappa^*, \Delta\} \\ &\quad + \frac{\hbar^2}{8}[\{\{\kappa, \Delta^*\}\} - \{\{\Delta, \kappa^*\}\}] + \dots, \\ -i\hbar\dot{\kappa} &= -h\kappa - \kappa\bar{h} - \frac{i\hbar}{2}\{h, \kappa\} - \frac{i\hbar}{2}\{\kappa, \bar{h}\} \\ &\quad + \Delta - \Delta\bar{f} - f\Delta - \frac{i\hbar}{2}\{f, \Delta\} - \frac{i\hbar}{2}\{\Delta, \bar{f}\} \\ &\quad + \frac{\hbar^2}{8}[\{\{h, \kappa\}\} + \{\{\kappa, \bar{h}\}\} + \{\{\Delta, \bar{f}\}\} + \{\{f, \Delta\}\}] + \dots, \\ -i\hbar\dot{\kappa}^* &= \kappa^*h + \bar{h}\kappa^* + \frac{i\hbar}{2}\{\kappa^*, h\} + \frac{i\hbar}{2}\{\bar{h}, \kappa^*\} \\ &\quad - \Delta^* + \bar{f}\Delta^* + \Delta^*f + \frac{i\hbar}{2}\{\bar{f}, \Delta^*\} + \frac{i\hbar}{2}\{\Delta^*, f\} \\ &\quad - \frac{\hbar^2}{8}[\{\{\kappa^*, h\}\} + \{\{\bar{h}, \kappa^*\}\} + \{\{\bar{f}, \Delta^*\}\} + \{\{\Delta^*, f\}\}] + \dots, \end{aligned} \quad (9)$$

where the functions h, f, Δ , and κ are the Wigner transforms of $\hat{h}, \hat{\rho}, \hat{\Delta}$, and $\hat{\kappa}$, respectively, $\bar{f}(\mathbf{r}, \mathbf{p}) = f(\mathbf{r}, -\mathbf{p})$, $\{f, g\}$ is the Poisson bracket of the functions $f(\mathbf{r}, \mathbf{p})$ and $g(\mathbf{r}, \mathbf{p})$ (see Appendix A); the dots stand for terms proportional to higher powers of \hbar .

To investigate collective modes described by these equations we apply the method of Wigner function moments. The idea of the method is based on the virial theorems by Chandrasekhar and Lebovitz [21]; its detailed formulation can be found in [22, 23]. To study the quadrupole collective motion in axially symmetric nuclei it is necessary to calculate moments of Eqs. (9) with the weight functions

$$xz, \quad p_x p_z, \quad zp_x + xp_z, \quad \text{and} \quad zp_x - xp_z. \quad (10)$$

This procedure means that we refrain from seeking the whole density matrix and restrict ourselves to the knowledge of only several moments. Nevertheless this information turns out to be sufficient for a satisfactory description of various collective modes with quantum numbers $K^\pi = 1^+$, as it was shown in our previous publications [1, 22, 23]. In the case without pairing, this restricted information can be extracted from the TDHF equations and becomes exact only for the harmonic oscillator with multipole-multipole residual interactions. For more realistic models it becomes approximate even without pairing. The TDHFB equations (9) are considerably more complicated than the TDHF ones, so additional approximations are necessary even for the simple model considered here. This is the subject of this section.

Let us at first write out several useful relations:

$$\begin{aligned} \int d^3p \int d^3r A\{f, g\} &= - \int d^3p \int d^3r f\{A, g\} = - \int d^3p \int d^3r g\{f, A\}, \\ \int d^3p \int d^3r A\{\{f, g\}\} &= \int d^3p \int d^3r f\{\{A, g\}\} = \int d^3p \int d^3r g\{\{f, A\}\}, \end{aligned}$$

where A is any one of the above mentioned weight functions, f and g are arbitrary functions and $\{\{f, g\}\}$ is defined in Appendix B. Integration of Eqs. (9) (including the terms of higher orders in \hbar) over the phase space with the weight A yields the following set of equations:

$$\begin{aligned} i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A f &= \int d(\mathbf{p}, \mathbf{r}) \left[i\hbar \{A, h\} f + A(\Delta^* \kappa - \kappa^* \Delta) - \frac{i\hbar}{2} (\{A, \Delta^*\} \kappa + \{A, \Delta\} \kappa^*) \right. \\ &\quad \left. - \frac{\hbar^2}{8} (\{\{A, \Delta^*\}\} \kappa - \{\{A, \Delta\}\} \kappa^*) \right], \\ i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \bar{f} &= \int d(\mathbf{p}, \mathbf{r}) \left[i\hbar \{\bar{h}, A\} \bar{f} + A(\Delta^* \kappa - \kappa^* \Delta) + \frac{i\hbar}{2} (\{A, \Delta^*\} \kappa + \{A, \Delta\} \kappa^*) \right. \\ &\quad \left. - \frac{\hbar^2}{8} (\{\{A, \Delta^*\}\} \kappa - \{\{A, \Delta\}\} \kappa^*) \right], \end{aligned}$$

$$\begin{aligned}
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \kappa &= \int d(\mathbf{p}, \mathbf{r}) \left[A(h + \bar{h})\kappa + \frac{i\hbar}{2} \{A, (h - \bar{h})\} \kappa - A\Delta(1 - \bar{f} - f) \right. \\
&\quad \left. + \frac{i\hbar}{2} \{A, \Delta\}(\bar{f} - f) - \frac{\hbar^2}{8} [\{\{A, (h + \bar{h})\}\} \kappa + \{\{A, \Delta\}\}(\bar{f} + f)] \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \kappa^* &= \int d(\mathbf{p}, \mathbf{r}) \left[-A(h + \bar{h})\kappa^* + \frac{i\hbar}{2} \{A, (h - \bar{h})\} \kappa^* + A\Delta^*(1 - \bar{f} - f) \right. \\
&\quad \left. + \frac{i\hbar}{2} \{A, \Delta^*\}(\bar{f} - f) + \frac{\hbar^2}{8} [\{\{A, (h + \bar{h})\}\} \kappa^* + \{\{A, \Delta^*\}\}(\bar{f} + f)] \right], \quad (11)
\end{aligned}$$

where $\int d(\mathbf{p}, \mathbf{r}) \equiv 2(2\pi\hbar)^{-3} \int d^3p \int d^3r$. It is necessary to note an essential point: there are no terms with higher powers of \hbar in these equations. The infinite number of terms proportional to \hbar^n with $n > 2$ have disappeared after integration, as is demonstrated in Appendix B. This fact does not mean, that higher powers of \hbar are not necessary for the exact solution of the problem. As it will be shown below, the set of equations (11) contains terms, which couple with dynamical equations of higher order moments, which include, naturally, the higher powers of \hbar .

It is convenient to rewrite the above equations in terms of $h_{\pm} = h \pm \bar{h}$, $f_{\pm} = f \pm \bar{f}$, $\Delta_{\pm} = \Delta \pm \Delta^*$, $\kappa_{\pm} = \kappa \pm \kappa^*$:

$$\begin{aligned}
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A f_+ &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} (\{A, h_+\} f_- + \{A, h_-\} f_+) + A(\Delta_+ \kappa_- - \kappa_+ \Delta_-) \right. \\
&\quad \left. - \frac{\hbar^2}{8} (\{\{A, \Delta_+\}\} \kappa_- - \{\{A, \Delta_-\}\} \kappa_+) \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A f_- &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} (\{A, h_+\} f_+ + \{A, h_-\} f_-) \right. \\
&\quad \left. - \frac{i\hbar}{2} (\{A, \Delta_+\} \kappa_+ - \{A, \Delta_-\} \kappa_-) \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \kappa_+ &= \int d(\mathbf{p}, \mathbf{r}) \left[A h_+ \kappa_- + \frac{i\hbar}{2} \{A, h_-\} \kappa_+ - A \Delta_- (1 - f_+) \right. \\
&\quad \left. - \frac{i\hbar}{2} \{A, \Delta_+\} f_- - \frac{\hbar^2}{8} (\{\{A, h_+\}\} \kappa_- + \{\{A, \Delta_-\}\} f_+) \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \kappa_- &= \int d(\mathbf{p}, \mathbf{r}) \left[A h_+ \kappa_+ + \frac{i\hbar}{2} \{A, h_-\} \kappa_- - A \Delta_+ (1 - f_+) \right. \\
&\quad \left. - \frac{i\hbar}{2} \{A, \Delta_-\} f_- - \frac{\hbar^2}{8} (\{\{A, h_+\}\} \kappa_+ + \{\{A, \Delta_+\}\} f_+) \right]. \quad (12)
\end{aligned}$$

These equations are strongly nonlinear, because Δ is a function of κ (see, e.g., ref. [20]):

$$\Delta(\mathbf{r}, \mathbf{p}) = \int \frac{d^3 p'}{(2\pi\hbar)^3} v(|\mathbf{p} - \mathbf{p}'|) \kappa(\mathbf{r}, \mathbf{p}'). \quad (13)$$

Having in mind small amplitude oscillations we will linearize: $f_{\pm} = f_{\pm}^0 + \delta f_{\pm}$, $\kappa_{\pm} = \kappa_{\pm}^0 + \delta \kappa_{\pm}$, $\Delta_{\pm} = \Delta_{\pm}^0 + \delta \Delta_{\pm}$. The Hamiltonian should be divided into the ground state Hamiltonian h^0 and the residual interaction h^1 (and, if necessary, the external field). We consider h^0 without \mathbf{p} -odd terms, hence $h_-^0 = 0$ and as a consequence $f_-^0 = 0$. It is natural to take Δ^0 real, i.e. $\Delta_-^0 = 0$, $\kappa_-^0 = 0$. Linearizing (12) and taking into account the last remarks we arrive at

$$\begin{aligned} i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta f_+ &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} (\{A, h_+^0\} \delta f_- + \{A, h_-^1\} f_+^0) + A(\Delta_+^0 \delta \kappa_- - \kappa_+^0 \delta \Delta_-) \right. \\ &\quad \left. - \frac{\hbar^2}{8} (\{\{A, \Delta_+^0\}\} \delta \kappa_- - \{\{A, \delta \Delta_-\}\} \kappa_+^0) \right], \\ i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta f_- &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} (\{A, h_+^0\} \delta f_+ + \{A, h_+^1\} f_+^0) \right. \\ &\quad \left. - \frac{i\hbar}{2} (\{A, \Delta_+^0\} \delta \kappa_+ + \{A, \delta \Delta_+\} \kappa_+^0) \right], \\ i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta \kappa_+ &= \int d(\mathbf{p}, \mathbf{r}) \left[A h_+^0 \delta \kappa_- + \frac{i\hbar}{2} \{A, h_-^1\} \kappa_+^0 - A \delta \Delta_- (1 - f_+^0) \right. \\ &\quad \left. - \frac{i\hbar}{2} \{A, \Delta_+^0\} \delta f_- - \frac{\hbar^2}{8} (\{\{A, h_+^0\}\} \delta \kappa_- + \{\{A, \delta \Delta_-\}\} f_+^0) \right], \\ i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta \kappa_- &= \int d(\mathbf{p}, \mathbf{r}) \left[A h_+^0 \delta \kappa_+ + A h_+^1 \kappa_+^0 - A \delta \Delta_+ (1 - f_+^0) + A \Delta_+^0 \delta f_+ \right. \\ &\quad \left. - \frac{\hbar^2}{8} (\{\{A, h_+^0\}\} \delta \kappa_+ + \{\{A, h_+^1\}\} \kappa_+^0 + \{\{A, \Delta_+^0\}\} \delta f_+ + \{\{A, \delta \Delta_+\}\} f_+^0) \right]. \quad (14) \end{aligned}$$

Until this point, our formulation is completely general.

Now let us consider the popular case of pure monopole pairing. This means that the variation of the gap,

$$\delta \Delta(\mathbf{r}, \mathbf{p}) = \int \frac{d^3 p'}{(2\pi\hbar)^3} v(|\mathbf{p} - \mathbf{p}'|) \delta \kappa(\mathbf{r}, \mathbf{p}'), \quad (15)$$

will be projected on its monopole part. In the case of quadrupole vibrations, which we will study here, the variations δf_{\pm} and $\delta \kappa_{\pm}$ will have quadrupole multipolarities. As a consequence, when projecting formula (15) on the monopole part, the integral over angles will be equal to zero and we get $\delta \Delta = 0$. We also note that neglecting $\delta \Delta$ corresponds to the usual Inglis

approximation. Then Eqs. (14) are reduced to

$$\begin{aligned}
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta f_+ &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} (\{A, h_+^0\} \delta f_- + \{A, h_-^1\} f_+^0) \right. \\
&\quad \left. + A \Delta_+^0 \delta \kappa_- - \frac{\hbar^2}{8} \{\{A, \Delta_+^0\}\} \delta \kappa_- \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta f_- &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} (\{A, h_+^0\} \delta f_+ + \{A, h_+^1\} f_+^0) - \frac{i\hbar}{2} \{A, \Delta_+^0\} \delta \kappa_+ \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta \kappa_+ &= \int d(\mathbf{p}, \mathbf{r}) \left[A h_+^0 \delta \kappa_- + \frac{i\hbar}{2} \{A, h_-^1\} \kappa_+^0 \right. \\
&\quad \left. - \frac{i\hbar}{2} \{A, \Delta_+^0\} \delta f_- - \frac{\hbar^2}{8} \{\{A, h_+^0\}\} \delta \kappa_- \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta \kappa_- &= \int d(\mathbf{p}, \mathbf{r}) \left[A h_+^0 \delta \kappa_+ + A h_+^1 \kappa_+^0 + A \Delta_+^0 \delta f_+ \right. \\
&\quad \left. - \frac{\hbar^2}{8} (\{\{A, h_+^0\}\} \delta \kappa_+ + \{\{A, h_+^1\}\} \kappa_+^0 + \{\{A, \Delta_+^0\}\} \delta f_+) \right]. \quad (16)
\end{aligned}$$

To proceed further we are forced to do two approximations to get rid of higher rank moments and obtain a closed set of dynamical equations for second rank moments. First, the integrals $\int d(\mathbf{p}, \mathbf{r}) A h_+^0 \delta \kappa_{\pm}$ contain fourth rank moments. The analysis of the integrand shows that we can neglect these integrals without a strong loss of accuracy. Indeed, the functions κ_{\pm} (and their variations) are sharply peaked at the Fermi surface, where the Hamiltonian h_+^0 by definition is equal to zero. Therefore the product $h_+^0 \delta \kappa_{\pm}$ should be small. Second, the (\mathbf{r}, \mathbf{p}) -dependence of Δ_+^0 can generate, in principle, an infinite number of moments of various ranks. To simplify the problem we will consider here the commonly employed approximation of an (\mathbf{r}, \mathbf{p}) -independent gap $\Delta_+^0 \equiv 2\Delta = \text{const}$, an approximation often used in nuclear physics and consistent with the monopole-monopole pairing force model. So, adding the isospin index $\tau = (p, n)$, we finally have

$$\begin{aligned}
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta f_+^\tau &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} (\{A, h_+^{\tau 0}\} \delta f_-^\tau + \{A, h_-^{\tau 1}\} f_+^{\tau 0}) + 2A \Delta^\tau \delta \kappa_-^\tau \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta f_-^\tau &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} (\{A, h_+^{\tau 0}\} \delta f_+^\tau + \{A, h_+^{\tau 1}\} f_+^{\tau 0}) \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta \kappa_+^\tau &= \int d(\mathbf{p}, \mathbf{r}) \left[\frac{i\hbar}{2} \{A, h_-^{\tau 1}\} \kappa_+^{\tau 0} - \frac{\hbar^2}{8} \{\{A, h_+^{\tau 0}\}\} \delta \kappa_-^\tau \right], \\
i\hbar \frac{d}{dt} \int d(\mathbf{p}, \mathbf{r}) A \delta \kappa_-^\tau &= \int d(\mathbf{p}, \mathbf{r}) \left[A h_+^{\tau 1} \kappa_+^{\tau 0} + 2A \Delta^\tau \delta f_+^\tau \right]
\end{aligned}$$

$$-\frac{\hbar^2}{8}(\{\{A, h_+^{\tau 0}\}\}\delta\kappa_+^\tau + \{\{A, h_+^{\tau 1}\}\}\kappa_+^{\tau 0}) \Big]. \quad (17)$$

We consider an axially symmetric model with $h_0^\tau = \frac{p^2}{2m} + \frac{1}{2}m[\omega_x^{\tau 2}(x^2 + y^2) + \omega_z^{\tau 2}z^2] - \mu^\tau$ and $h_1^\tau = Z^\tau(t)xz$ with $Z^n = \chi_{nn}Q^n + \chi_{np}Q^p$ and $Z^p = \chi_{pp}Q^p + \chi_{np}Q^n$, μ^τ being the chemical potential of either protons ($\tau=p$) or neutrons ($\tau=n$). $\chi_{\tau\tau'}$ is the strength constant of the quadrupole-quadrupole residual interaction, Q^τ is a component of the quadrupole moment $Q^\tau(t) = \int d(\mathbf{p}, \mathbf{r})xz\delta f^\tau(\mathbf{r}, \mathbf{p}, t)$. It is supposed that $\chi_{nn} = \chi_{pp}$. Obviously, in this model $h_-^\tau = 0$ and $h_+^\tau = 2h^\tau$. Calculating the required Poisson brackets

$$\begin{aligned} \{xz, h_0^\tau\} &= \frac{1}{m}\hat{L}, \quad \{p_x p_z, h_0^\tau\} = -m(\omega_z^{\tau 2}zp_x + \omega_x^{\tau 2}xp_z), \quad \{zp_x, h_0^\tau\} = \frac{1}{m}p_x p_z - m\omega_x^{\tau 2}xz, \\ \{xp_z, h_0^\tau\} &= \frac{1}{m}p_x p_z - m\omega_z^{\tau 2}xz, \quad \{xz, h_1^\tau\} = 0, \quad \{p_x p_z, h_1^\tau\} = -Z^\tau(t)(zp_x + xp_z), \\ \{zp_x, h_1^\tau\} &= -Z^\tau(t)z^2, \quad \{xp_z, h_1^\tau\} = -Z^\tau(t)x^2, \quad \{\{A, h_0^\tau\}\} = 0, \\ \{\{xz, h_1^\tau\}\} &= 0, \quad \{\{p_x p_z, h_1^\tau\}\} = 2Z^\tau(t), \quad \{\{zp_x, h_1^\tau\}\} = 0, \quad \{\{xp_z, h_1^\tau\}\} = 0. \end{aligned}$$

we find, that the third equation of (17) becomes trivial, giving four integrals of motion $\int d(\mathbf{p}, \mathbf{r})A\delta\kappa_+^\tau$. Introducing the notation

$$\begin{aligned} Q^\tau(t) &= \int d(\mathbf{p}, \mathbf{r})xz\delta f^\tau(\mathbf{r}, \mathbf{p}, t), \quad \tilde{Q}^\tau(t) = \int d(\mathbf{p}, \mathbf{r})xz\delta\kappa_-^\tau(\mathbf{r}, \mathbf{p}, t), \\ P^\tau(t) &= \int d(\mathbf{p}, \mathbf{r})p_x p_z\delta f^\tau(\mathbf{r}, \mathbf{p}, t), \quad \tilde{P}^\tau(t) = \int d(\mathbf{p}, \mathbf{r})p_x p_z\delta\kappa_-^\tau(\mathbf{r}, \mathbf{p}, t), \\ L^\tau(t) &= \int d(\mathbf{p}, \mathbf{r})(zp_x + xp_z)\delta f^\tau(\mathbf{r}, \mathbf{p}, t), \quad \tilde{L}^\tau(t) = \int d(\mathbf{p}, \mathbf{r})(zp_x + xp_z)\delta\kappa_-^\tau(\mathbf{r}, \mathbf{p}, t), \\ I_y^\tau(t) &= \int d(\mathbf{p}, \mathbf{r})(zp_x - xp_z)\delta f^\tau(\mathbf{r}, \mathbf{p}, t), \quad \tilde{I}_y^\tau(t) = \int d(\mathbf{p}, \mathbf{r})(zp_x - xp_z)\delta\kappa_-^\tau(\mathbf{r}, \mathbf{p}, t) \end{aligned}$$

we find the following set of dynamical equations

$$\begin{aligned} i\hbar\dot{Q}^\tau &= i\hbar\frac{1}{m}L^\tau + \Delta^\tau\tilde{Q}^\tau, \\ i\hbar\dot{\tilde{Q}}^\tau &= 4\Delta^\tau Q^\tau + 2k_4^\tau Z^\tau, \\ i\hbar\dot{P}^\tau &= -i\hbar\frac{m}{2}[(\omega_x^{\tau 2} + \omega_z^{\tau 2})L^\tau - (\omega_x^{\tau 2} - \omega_z^{\tau 2})I_y^\tau] + \Delta^\tau\tilde{P}^\tau, \\ i\hbar\dot{\tilde{P}}^\tau &= 4\Delta^\tau P^\tau - \frac{\hbar^2}{2}k_0^\tau Z^\tau, \\ i\hbar\dot{L}^\tau &= i\hbar[\frac{2}{m}P^\tau - m(\omega_x^{\tau 2} + \omega_z^{\tau 2})Q^\tau - (\langle z^2 \rangle^\tau + \langle x^2 \rangle^\tau)Z^\tau], \\ i\hbar\dot{I}_y^\tau &= i\hbar m(\omega_z^{\tau 2} - \omega_x^{\tau 2})Q^\tau - i\hbar(\langle z^2 \rangle^\tau - \langle x^2 \rangle^\tau)Z^\tau. \end{aligned} \quad (18)$$

where

$$\begin{aligned}\langle x^2 \rangle^\tau &= \int d(\mathbf{p}, \mathbf{r}) x^2 f^{\tau 0}(\mathbf{r}, \mathbf{p}), & \langle z^2 \rangle^\tau &= \int d(\mathbf{p}, \mathbf{r}) z^2 f^{\tau 0}(\mathbf{r}, \mathbf{p}), \\ k_0^\tau &= \int d(\mathbf{p}, \mathbf{r}) \kappa_+^{\tau 0}(\mathbf{r}, \mathbf{p}), & k_4^\tau &= \int d(\mathbf{p}, \mathbf{r}) x^2 z^2 \kappa_+^{\tau 0}(\mathbf{r}, \mathbf{p}).\end{aligned}\quad (19)$$

Eqs. (18) will be simplified as far as possible to obtain results in analytical form.

3 Simplified model

The scissors mode is an isovector one, so it is natural to rewrite Eqs. (18) in isoscalar and isovector terms. For the scissors mode, which we are interested in, the isovector set of equations can be decoupled from the isoscalar one with the help of the following approximations:

$$\begin{aligned}\Delta^p &\simeq \Delta^n = \Delta, & k_4^p &\simeq k_4^n = k_4/2, & k_0^p &\simeq k_0^n = k_0/2, \\ \langle x^2 \rangle^p &\simeq \langle x^2 \rangle^n = \langle x^2 \rangle/2, & \langle z^2 \rangle^p &\simeq \langle z^2 \rangle^n = \langle z^2 \rangle/2, & \delta^p &\simeq \delta^n = \delta,\end{aligned}$$

where δ is the nucleus deformation. Introducing isovector variables $\underline{Q} = Q^n - Q^p$, $\underline{\tilde{Q}} = \tilde{Q}^n - \tilde{Q}^p$ and so on, we can write the isovector set of equations as

$$\begin{aligned}i\hbar \dot{\underline{Q}} &= i\hbar \frac{1}{m} \underline{L} + \Delta \underline{\tilde{Q}}, \\ i\hbar \dot{\underline{\tilde{Q}}} &= 4\Delta \underline{Q} + 2k_4 \chi_1 \underline{Q}, \\ i\hbar \dot{\underline{P}} &= -i\hbar m \bar{\omega}^2 \left[\left(1 + \frac{\delta}{3}\right) \underline{L} - \delta \underline{I}_y \right] + \Delta \underline{\tilde{P}}, \\ i\hbar \dot{\underline{\tilde{P}}} &= 4\Delta \underline{P} - \frac{\hbar^2}{2} k_0 \chi_1 \underline{Q}, \\ i\hbar \dot{\underline{L}} &= i\hbar \left[\frac{2}{m} \underline{P} - (2m\bar{\omega}^2 + \frac{2}{3} Q_{00} \chi_1) \left(1 + \frac{\delta}{3}\right) \underline{Q} \right], \\ i\hbar \dot{\underline{I}_y} &= -i\hbar \delta (2m\bar{\omega}^2 + \frac{2}{3} Q_{00} \chi_1) \underline{Q},\end{aligned}\quad (20)$$

where $\chi_1 = \frac{1}{2}(\chi_{nn} - \chi_{np})$ is the isovector strength constant. Usually one takes $\chi_1 = \alpha \chi_0$, α being a fitting parameter. For the isoscalar strength constant χ_0 we will take the self consistent value

$$\chi_0 = -\frac{3m\bar{\omega}^2}{Q_{00}}. \quad (21)$$

Following ref. [1] we take $\alpha = -2$, i.e. a repulsive interaction with magnitude twice as large as the isoscalar one. Deriving (20) we used the self consistent expressions for the oscillator frequencies [1]

$$\omega_x^2 = \omega_y^2 = \bar{\omega}^2(1 + \frac{4}{3}\delta), \quad \omega_z^2 = \bar{\omega}^2(1 - \frac{2}{3}\delta), \quad \bar{\omega}^2 = \omega^2/(1 + \frac{2}{3}\delta) \quad (22)$$

and the standard [24] definition of the deformation parameter $\delta = 3Q_{20}/(4Q_{00})$, where $Q_{00} = 2 \int d(\mathbf{p}, \mathbf{r})(x^2 + y^2 + z^2)f^0(\mathbf{r}, \mathbf{p}) = 2\langle x^2 \rangle + \langle z^2 \rangle$ and $Q_{20} = 2\langle z^2 \rangle - 2\langle x^2 \rangle$ are monopole and quadrupole moments of nucleus, respectively.

This set of equations has two integrals of motion :

$$\frac{i\hbar}{2m\Delta}\tilde{P} + [\frac{\hbar^2\chi_1k_0}{4m\Delta} - 2m\bar{\omega}^2(1 - \alpha)(1 + \frac{\delta}{3})]\frac{i\hbar}{4\Delta + 2\chi_1k_4}\tilde{Q} - \underline{L} = const \quad (23)$$

and

$$\underline{I}_y + 2m\bar{\omega}^2\delta(1 - \alpha)\frac{i\hbar}{4\Delta + 2\chi_1k_4}\tilde{Q} = const. \quad (24)$$

Obviously these constants should be equal zero. By definition the variable \tilde{Q} is purely imaginary because κ_- is the imaginary part of the pairing field κ . Therefore Eq. (24) implies that the relative angular momentum \underline{I}_y oscillates in phase with the relative quadrupole moment \tilde{Q} of the imaginary part of the pairing field κ .

Analogously one can interpret Eq. (23) saying, that the variable \underline{L} oscillates out of phase with the linear combination of two variables \tilde{Q} and \tilde{P} which describe the quadrupole deformation of the pairing field in coordinate and momentum spaces respectively.

3.1 Eigenfrequencies

Imposing the time evolution via $e^{i\Omega t}$ for all variables one transforms Eqs. (20) into a set of algebraic equations, whose determinant gives the eigenfrequencies of the system. We have

$$Det = (\mathcal{E}^2 - 2\chi_1k_4\Delta)(\mathcal{E}^2 - 2\epsilon^2) - 2\epsilon^2(1 - \alpha)\mathcal{E}^2 - \chi_1\Delta k_0\hbar^4/m^2 + 4\hbar^4\bar{\omega}^4(1 - \alpha)\delta^2 = 0, \quad (25)$$

where $\mathcal{E}^2 = E^2 - 4\Delta^2$, $E = \hbar\Omega$, $\epsilon^2 = \hbar^2\bar{\omega}^2(1 + \frac{\delta}{3})$.

In the case of $\Delta = 0$ this equation is reduced to the known [1, 25] equation for the scissors mode. In the case $\Delta \neq 0$ there are two solutions:

$$E_{\pm}^2 = 4\Delta^2 + [\epsilon^2(2 - \alpha) + \chi_1k_4\Delta]$$

$$\pm\sqrt{[\epsilon^2(2-\alpha) + \chi_1 k_4 \Delta]^2 - \chi_1 \Delta[4k_4 \epsilon^2 - k_0 \hbar^4/m^2] - 4\hbar^4 \bar{\omega}^4(1-\alpha)\delta^2}. \quad (26)$$

They describe the energies of the isovector GQR (E_+) and of the scissors mode (E_-).

It is worth noting that contrary to the case without pairing [1] the energy of the scissors mode does not go to zero for deformation $\delta = 0$. However this does not mean any contradiction with the known quantum mechanical statement that the rotation of spherical nuclei is impossible. It is easy to see from (24) that the relative angular momentum \underline{I}_y is conserved in this case, $\underline{I}_y = \text{const}$, so the nature of this mode of a spherical nucleus has nothing in common with the vibration of angular momentum. The calculation of transition probabilities (see below) shows that this mode can be excited by an electric field and it is not excited by a magnetic field. Our estimation gives for the energy of this mode the value about 4 MeV, that agrees very well with the result of M. Matsuo et al [26], who studied the isovector quadrupole response of ^{158}Sn in the framework of QRPA with Skyrme forces and found the proper resonance at ~ 4.2 MeV.

It is known [1, 2] that without pairing the scissors mode has a non zero value of an energy only due to the Fermi Surface Deformation (FSD). Let us investigate the role of FSD in the case with pairing. Omitting in (20) the variable \underline{P} responsible for FSD and its dynamical equation we obtain the following characteristic equation

$$E^2[E^2 - 4\Delta^2 - 2\alpha\chi_0 k_4 \Delta - 2\epsilon^2(1-\alpha)] = 0.$$

Two solutions $E_{sc}^2 = 0$ and $E_{iv}^2 = 4\Delta^2 + 2\alpha\chi_0 k_4 \Delta + 2\epsilon^2(1-\alpha)$ reproduce the situation observed without pairing: the role of FSD is crucial for the scissors mode and is not very important for IVGQR.

3.2 Transition probabilities

The transition probabilities are calculated with the help of the linear response theory. The detailed description of its use within the framework of WFM method can be found in [1], so we only present the final results.

Electric quadrupole excitations are described by the operator

$$\hat{F} = \hat{F}_{2\mu}^{\text{p}} = \sum_{s=1}^Z \hat{f}_{2\mu}(s), \quad \hat{f}_{2\mu} = e r^2 Y_{2\mu}. \quad (27)$$

The transition probabilities are

$$B(E2)_\nu = 2 | \langle \nu | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar^2}{m} \frac{5}{16\pi} Q_{00} \frac{(1 + \delta/3)(E_\nu^2 - 4\Delta^2) - 2(\hbar\bar{\omega}\delta)^2}{E_\nu[E_\nu^2 - 4\Delta^2 - \epsilon^2(2 - \alpha) - \chi_1 k_4 \Delta]}. \quad (28)$$

Magnetic dipole excitations are described by the operator

$$\hat{F} = \hat{F}_{1\mu}^p = \sum_{s=1}^Z \hat{f}_{1\mu}(s), \quad \hat{f}_{1\mu} = -i\nabla(rY_{1\mu}) \cdot [\mathbf{r} \times \nabla] \mu_N, \quad \mu_N = \frac{e\hbar}{2mc}. \quad (29)$$

For transition probabilities we have

$$B(M1)_\nu = 2 | \langle \nu | \hat{F}_{11}^p | 0 \rangle |^2 = \frac{1 - \alpha}{8\pi} m\bar{\omega}^2 Q_{00} \delta^2 \frac{E_\nu^2 - 4\Delta^2 - 2\epsilon^2}{E_\nu[E_\nu^2 - 4\Delta^2 - \epsilon^2(2 - \alpha) - \chi_1 k_4 \Delta]} \mu_N^2. \quad (30)$$

Multiplying B(M1) factors of both states by the proper energies and summing we find the following formula for the energy weighted sum rule

$$E_{sc} B(M1)_{sc} + E_{iv} B(M1)_{iv} = (1 - \alpha) \frac{m\bar{\omega}^2}{4\pi} Q_{00} \delta^2 \mu_N^2. \quad (31)$$

This expression coincides exactly with the respective sum rule calculated in [1] without pairing. This means that there is no contribution to the sum rule which comes from pairing. This result can be explained by our approximation $\Delta = \text{const}$.

It is now a good place to discuss the deformation dependence of the energies and transition probabilities. First we recall the corresponding formulae without pairing:

$$\begin{aligned} (E_{iv}^0)^2 &= 4\hbar^2 \bar{\omega}^2 \left(1 + \frac{\delta}{3} + \sqrt{\left(1 + \frac{\delta}{3}\right)^2 - \frac{3}{4}\delta^2} \right), \\ (E_{sc}^0)^2 &= 4\hbar^2 \bar{\omega}^2 \left(1 + \frac{\delta}{3} - \sqrt{\left(1 + \frac{\delta}{3}\right)^2 - \frac{3}{4}\delta^2} \right), \\ B(M1)_\nu^0 &= \frac{3}{8\pi} m\bar{\omega}^2 Q_{00} \delta^2 \frac{E_\nu^2 - 2\epsilon^2}{E_\nu(E_\nu^2 - 4\epsilon^2)} \mu_N^2, \end{aligned} \quad (32)$$

where the superscript “0” means the absence of pairing and we assumed $\alpha = -2$. The scissors mode energy is proportional to δ , that becomes evident after expanding the square root:

$$E_{sc}^0 = \hbar\bar{\omega}\delta \sqrt{\frac{3}{2\delta_3}} \left(1 + \frac{3}{16} \frac{\delta^2}{\delta_3^2} + \frac{9}{128} \frac{\delta^4}{\delta_3^4} + \dots \right), \quad (33)$$

where $\delta_3 = 1 + \delta/3$.

At a first superficial glance, the transition probability, as given by formula (32), has the desired (experimentally observed) quadratic deformation dependence. However, due to the linear δ -dependence of the factor E_{sc} in the denominator, the resulting δ -dependence of $B(M1)_{sc}^0$ turns out to be linear, too. The situation is changed radically when pairing is included. In this case the main contribution to the scissors mode energy comes from the pairing interaction (the term $4\Delta^2$ in (26)), E_{sc} is not proportional to δ and the deformation dependence of $B(M1)_{sc}$ becomes quadratic in excellent agreement with QRPA calculations and experimental data [2, 4, 16, 18, 25, 27, 28, 29].

The deformation dependence of $B(M1)_{iv}$ is quadratic in δ , even without pairing, because the energy E_{iv} is not proportional to δ and depends only weakly on it. The pairing does not change this picture.

4 Numerical results and discussion

We have reproduced all experimentally observed qualitative features of the scissors mode. We understand that our model is too simplified to describe also precise quantitative experimental characteristics. Nevertheless we performed the calculations of energies and $B(M1)$ factors to get at least an idea on the order of magnitude of the discrepancy with experimental data. The results of calculations for most nuclei, where this mode is observed, are presented in Table 1 and in Figures 1–4. Formulae (26) and (30) were used with the following values of parameters: $\alpha = -2$, $Q_{00} = A^{\frac{3}{5}}R^2$, $R = r_0A^{1/3}$, $r_0 = 1.2$ fm, $\bar{\omega}^2 = \omega_0^2/[(1 + \frac{4}{3}\delta)^{2/3}(1 - \frac{2}{3}\delta)^{1/3}]$, $\hbar\omega_0 = 41/A^{1/3}$ MeV, $\hbar^2/m = 41.803$ MeV fm². The gap Δ as well as the integrals k_0 , k_4 were calculated by two methods: the semiclassical one, which is summarized in Appendix C, and the microscopical one described in Appendix D.

Let us analyse at first the results obtained with semiclassical values of Δ , k_0 , k_4 (columns *th* and *I* of Table 1). This method gives $\Delta \approx 1.32$ MeV for all the nuclei considered here. The values of k_0 , k_4 vary smoothly from $k_0 = 37.7$, $k_4 = 4243.7$ fm⁴ for A=134 to $k_0 = 54.7$, $k_4 = 9946.3$ fm⁴ for A=196. Analysis of Table 1 shows, that overall agreement of theoretical results with experimental data is reasonable. It is, of course, not perfect, but the influence of pairing, especially on $B(M1)$ values, is impressive. As it is seen, without pairing the calculated

Table 1: Scissors mode energies E_{sc} (in MeV) and transition probabilities $B(M1)_{sc}$ (in units of μ_N^2); *exp*: experimental values, *th1* and *th*: full theory with k_0, k_4 calculated by microscopical and semiclassical methods respectively, I: $\Delta \neq 0$ but $k_0 = k_4 = 0$, II: theory without pairing (i.e. $\Delta = k_0 = k_4 = 0$). The experimental values of E_{sc} , δ and $B(M1)$ are from Ref. [17] and references therein. $E1$ is explained in section 5.

Nuclei	δ	E_1	E_{sc}					$B(M1)_{sc}$				
			<i>exp</i>	<i>th1</i>	<i>th</i>	I	II	<i>exp</i>	<i>th1</i>	<i>th</i>	I	II
^{134}Ba	0.14	2.56	2.99	3.67	3.94	2.93	1.28	0.56	1.22	1.16	1.72	3.90
^{144}Nd	0.11	2.07	3.21	3.18	3.86	2.83	1.04	0.17	1.05	0.86	1.30	3.54
^{146}Nd	0.13	2.35	3.47	3.58	3.91	2.89	1.18	0.72	1.21	1.13	1.69	4.14
^{148}Nd	0.17	2.96	3.37	3.92	4.02	3.02	1.48	0.78	1.78	1.79	2.65	5.39
^{150}Nd	0.22	3.83	3.04	4.24	4.25	3.26	1.92	1.61	2.81	2.94	4.28	7.26
^{148}Sm	0.12	2.21	3.07	3.64	3.88	2.86	1.11	0.43	1.07	1.02	1.59	3.96
^{150}Sm	0.16	2.83	3.13	3.97	4.00	2.99	1.42	0.92	1.63	1.68	2.50	5.26
^{152}Sm	0.24	4.02	2.99	3.77	4.30	3.32	2.02	2.26	3.70	3.27	4.75	7.81
^{154}Sm	0.26	4.32	3.20	3.77	4.39	3.41	2.17	2.18	4.42	3.79	5.50	8.65
^{156}Gd	0.26	4.29	3.06	3.78	4.39	3.40	2.16	2.73	4.44	3.82	5.55	8.76
^{158}Gd	0.26	4.36	3.14	3.73	4.41	3.43	2.19	3.39	4.77	4.01	5.84	9.12
^{160}Gd	0.27	4.39	3.18	3.64	4.42	3.44	2.21	2.97	5.08	4.14	6.02	9.38
^{160}Dy	0.26	4.24	2.87	3.50	4.37	3.39	2.13	2.42	4.91	3.89	5.68	9.03
^{162}Dy	0.26	4.25	2.96	3.43	4.38	3.39	2.14	2.49	5.19	3.99	5.83	9.25
^{164}Dy	0.26	4.31	3.14	3.36	4.40	3.41	2.17	3.18	5.59	4.17	6.09	9.59
^{164}Er	0.25	4.17	2.90	3.85	4.35	3.37	2.10	1.45	4.40	3.94	5.77	9.26
^{166}Er	0.26	4.23	2.96	3.78	4.37	3.39	2.13	2.67	4.73	4.12	6.03	9.59
^{168}Er	0.26	4.18	3.21	3.70	4.36	3.37	2.10	2.82	4.85	4.11	6.04	9.67
^{170}Er	0.26	4.15	3.22	3.61	4.35	3.36	2.09	2.63	5.02	4.14	6.08	9.79
^{172}Yb	0.25	4.08	3.03	3.59	4.33	3.34	2.05	1.94	4.96	4.08	6.01	9.79
^{174}Yb	0.25	4.02	3.15	3.48	4.31	3.32	2.02	2.70	5.10	4.05	5.98	9.82
^{176}Yb	0.24	3.85	2.96	3.32	4.26	3.27	1.94	2.66	5.06	3.83	5.67	9.58
^{178}Hf	0.22	3.57	3.11	3.65	4.19	3.19	1.79	2.04	3.90	3.40	5.06	9.00
^{180}Hf	0.22	3.50	2.95	3.54	4.17	3.17	1.76	1.61	3.96	3.34	4.97	8.97
^{182}W	0.20	3.25	3.10	3.40	4.10	3.10	1.63	1.65	3.63	2.96	4.44	8.43
^{184}W	0.19	3.09	3.31	3.36	4.07	3.06	1.55	1.12	3.37	2.74	4.12	8.14
^{186}W	0.18	2.97	3.20	3.32	4.04	3.03	1.49	0.82	3.22	2.60	3.91	7.95
^{190}Os	0.15	2.42	2.90	3.35	3.93	2.90	1.21	0.98	2.15	1.82	2.77	6.64
^{192}Os	0.14	2.29	3.01	3.20	3.90	2.87	1.15	1.04	2.06	1.66	2.54	6.37
^{196}Pt	0.11	1.87	2.68	3.00	3.83	2.80	0.94	0.70	1.52	1.16	1.79	5.35

energies (column II) are 1 – 2 MeV (1.5 – 2 times) smaller, than E_{exp} , and $B(M1)$ factors (column II) exceed experimental values 3 – 7 times. Taking into account Δ (without k_4, k_0) changes the results (columns I) drastically: the differences between calculated and experimental energies are reduced to 5 – 10% and calculated transitions probabilities are reduced by a factor of 1.5 – 2. Inclusion of k_4 (columns *th*) reduces the transition probabilities again by a factor of 1.5, improving the agreement appreciably, and increases the energies by ~ 1 MeV deteriorating slightly the agreement with experimental data. The influence of k_0 is negligibly small, being of the order of $\sim 1\%$. Finally we obtain that E_{th} exceeds E_{exp} by $\sim 1 - 1.3$ MeV and that $B(M1)_{th}$ exceeds $B(M1)_{exp}$ approximately by a factor of 1.5 – 2.

Let us analyse now the results obtained with microscopical values of Δ , k_0 , k_4 (columns *th1* of Table 1). Microscopical gaps for neutrons and protons vary in the range 0.8 – 1.0 MeV and 1.0 – 1.2 MeV respectively, being appreciably smaller than the semiclassical ones. The values of k_4 vary in the limits 4888 – 8936 fm⁴, being in the reasonable agreement with semiclassical values (in general they are smaller by $\sim 10 - 20 \%$). Generally they increase with the atomic number A , however, due to shell effects it happens not monotonically, as in the semiclassical case, but quite irregularly. The integrals k_0 vary in the range 56 – 78. They are very sensitive to the details of level schemes – that is why they differ substantially (sometimes 60 – 70%) from the semiclassical values, being nevertheless of the same order of magnitude.

The analysis of final results (Table 1) reveals the following trend: the scissors mode energies decrease (in comparison with those of column *th*) by $\sim 0.5 - 0.7$ MeV, that improves the agreement with experimental data, and $B(M1)$ factors increase by $\sim 10 - 20 \%$, that leads to a somewhat worse agreement with experimental data.

What can be done to improve these results? The first step is obvious – it is necessary to get rid off approximations enumerated at the beginning of section 3, especially of the most crude one: $\Delta^p = \Delta^n$. As a result, it will be necessary to solve the coupled isoscalar and isovector sets of equations. The next possible step is to perform self consistent calculation with a more or less realistic interaction and taking into account the \mathbf{r} -dependence of Δ .

Another point, which should be clarified, is the role of the spin-orbit interaction. It is known [2], that experimentally observed low lying magnetic dipole strength consists of two separated

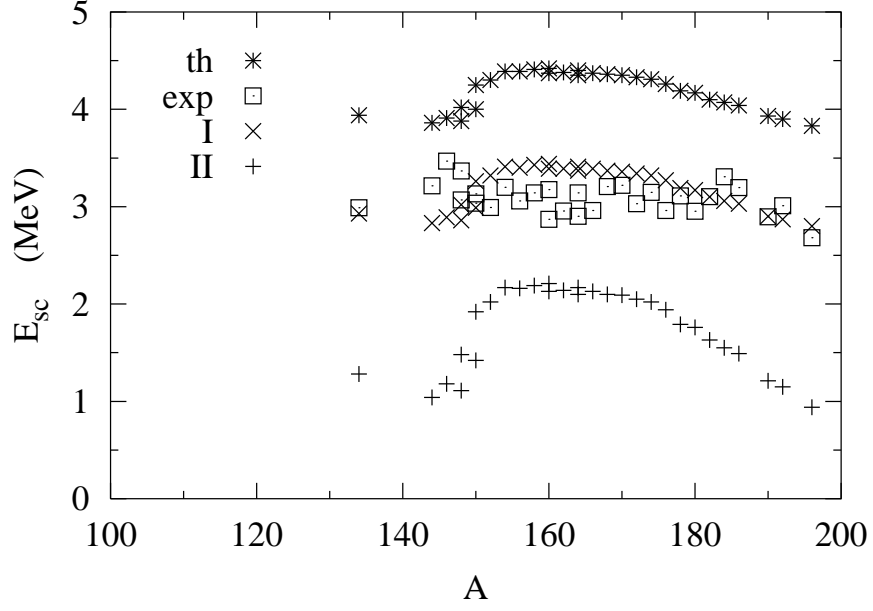


Figure 1: Scissors mode energies as a function of the mass number A for the nuclei listed in Table 1. The meaning of the symbols is explained in the caption of Table 1.

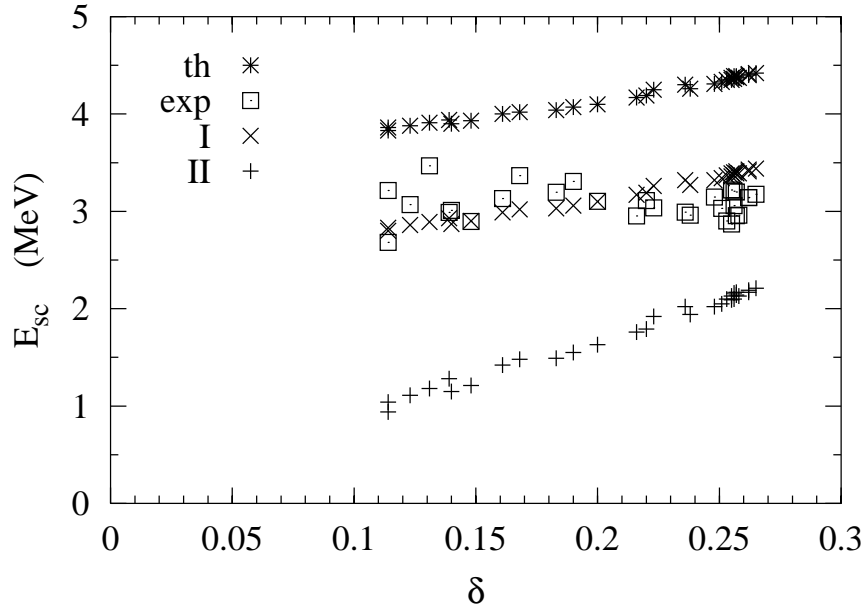


Figure 2: Scissors mode energies as a function of the deformation δ for the nuclei listed in Table 1. The meaning of the symbols is explained in the caption of Table 1.

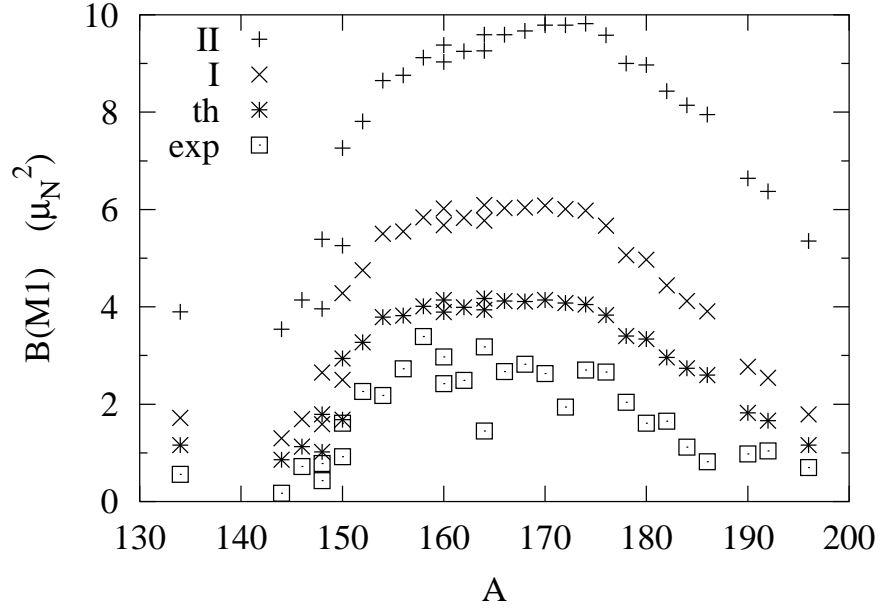


Figure 3: Scissors mode transition probabilities $B(M1)$ as a function of the mass number A for the nuclei listed in Table 1. The meaning of the symbols is explained in the caption of Table 1.

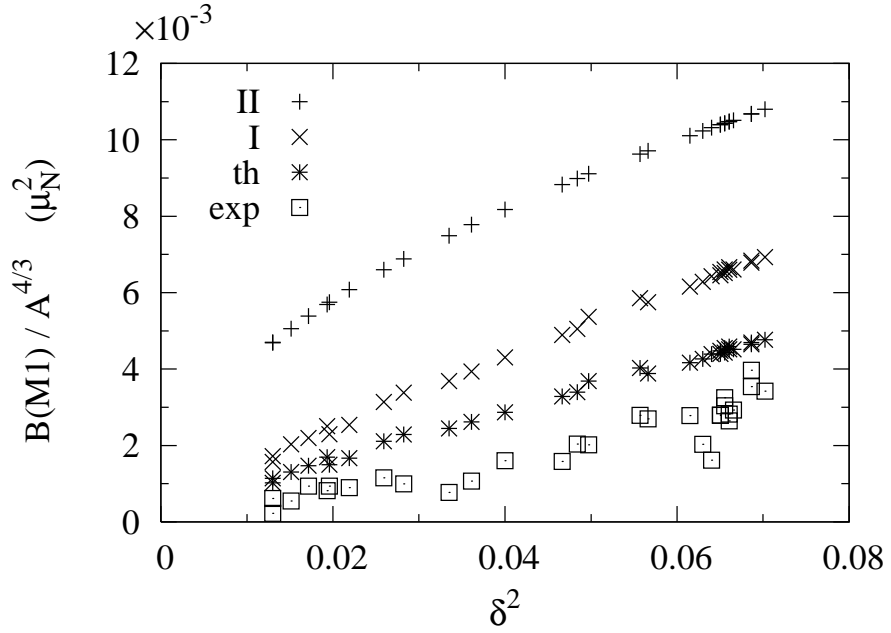


Figure 4: Scissors mode transition probabilities, normalized by a factor of $A^{-4/3}$, as a function of the deformation δ for the nuclei listed in Table 1. The meaning of the symbols is explained in the caption of Table 1.

parts: orbital excitations in an energy interval $\sim 2 - 4$ MeV and the spin-flip resonance ranging from 5 to 10 MeV excitation energy. So, for the full description of the scissors mode dynamics it would be necessary to consider also the spin degrees of freedom. One may expect that the orbital part of the M1 strength (scissors mode) will be pushed down by the spin-orbit interaction, in agreement with experimental data.

5 Currents

According [1] the components of infinitesimal displacements in the plane $y = 0$ are given by

$$\xi_x(t) = \sqrt{2}B\underline{Q}(t)z, \quad \xi_z(t) = \sqrt{2}A\underline{Q}(t)x \quad (34)$$

with

$$\begin{aligned} A &= \frac{3}{\sqrt{2}}[1 - 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta]/[Q_{00}(1 - \frac{2}{3}\delta)], \\ B &= \frac{3}{\sqrt{2}}[1 + 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta]/[Q_{00}(1 + \frac{4}{3}\delta)]. \end{aligned} \quad (35)$$

It is useful to write these displacement as the superposition of a rotational component with the coefficient a and an irrotational one with the coefficient b

$$\vec{\xi} = a\vec{e}_y \times \vec{r} + b\nabla(xz) = a(z, 0, -x) + b(z, 0, x) \quad \longrightarrow \quad \xi_x = (b + a)z, \quad \xi_z = (b - a)x. \quad (36)$$

Comparison of (34) with (36) gives

$$b + a = \sqrt{2}B\underline{Q}, \quad b - a = \sqrt{2}A\underline{Q} \quad \longrightarrow \quad b = (B + A)\underline{Q}/\sqrt{2}, \quad a = (B - A)\underline{Q}/\sqrt{2}.$$

Using here expressions (35) one finds

$$b = \gamma(\delta_3 - \delta^2/g), \quad a = \gamma\delta(1 - \delta_3/g), \quad (37)$$

where $g = E^2/(6\hbar^2\bar{\omega}^2)$, $\delta_3 = 1 + \delta/3$ and $\gamma = 3\underline{Q}/[O_{00}(1 - \frac{2}{3}\delta)(1 + \frac{4}{3}\delta)]$. The coefficients a, b are shown schematically as the functions of g on the figure 5. In spite of the simple behaviour of two curves, they describe rather interesting phenomena. There are two critical points: $g_1 = \delta^2/\delta_3$, where $b = 0$, and $g_2 = \delta_3$, where $a = 0$. They divide g axes into three regions.

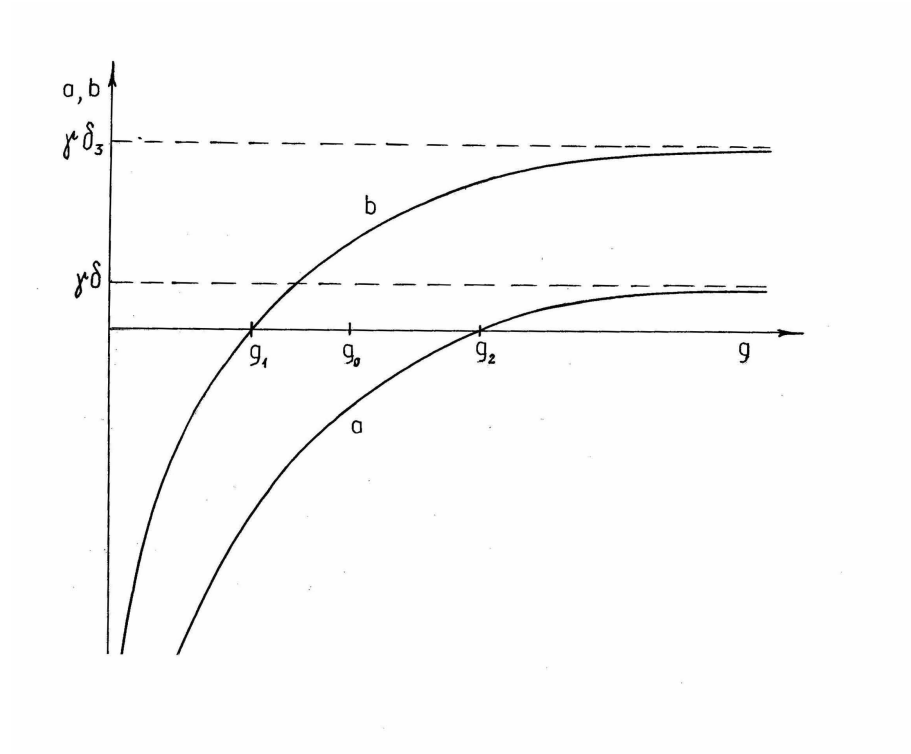


Figure 5: Amplitudes a and b (in arbitrary units) of the rotational and irrotational current components as the functions of energy ($g = E^2/(6\hbar^2\bar{\omega}^2)$).

In the region $0 < g < g_1$ the rotational component of the current dominates. The ratio b/a is changed from δ/δ_3 at $g = 0$ to zero at $g = g_1$. At this point the motion is pure rotational! The energy corresponding to this point is $E_1 = \hbar\bar{\omega}\delta\sqrt{6/\delta_3}$. Its value for various nuclei is shown in the Table 1. It is interesting that all experimental values of E_{sc} are disposed just in the vicinity of E_1 , the energies of nuclei with the small deformation ($\delta \leq .19$) being disposed on the right hand side of this point, while the energies of nuclei with large deformations ($\delta \geq .2$) are disposed on the left hand side. The theoretical value of E_{sc} obtained without pairing (see formula (33)) is $E_{sc}^0 \simeq \frac{1}{2}E_1$, i.e. it is disposed far to the left of E_1 for all nuclei independently of their deformations. Inclusion of pairing shifts the scissors mode energy E_{sc} strongly to the right so, that it becomes larger than the critical value E_1 , but quite close to it, especially in the well deformed nuclei. One can note, that the fact of the proximity of theoretical and, especially, experimental values of E_{sc} to the critical point E_1 gives some grounds for the two rotors model of the scissors mode [4], where the pure rotational motion of the nuclear matter is supposed.

In the region $g_1 < g < g_2$ the rotational and irrotational components of motion compete on an equal footing – the dominance of the rotational motion at the point $g = g_1$ is gradually replaced by the dominance of the irrotational motion at the point $g = g_2$, the strengths of both components being equal (i.e. $|a| = |b|$) at the point $g_0 = \delta$. The energy corresponding to the point g_2 is $E_2 = \sqrt{6\delta_3}\hbar\bar{\omega} = \sqrt{3\delta_3}\sqrt{2}\hbar\bar{\omega}$. The energy centroid of IVGQR (without pairing) is $E_{iv} = 2\sqrt{2}\hbar\bar{\omega}$. For normally deformed nuclei ($\delta < 1$) the factor $\sqrt{3\delta_3}$ is smaller than 2. Therefore E_{iv} is disposed on the right hand side of the point g_2 , i. e. in the region $g_2 < g < \infty$, where the irrotational component of the current dominates. The inclusion of pairing shifts E_{iv} more to the right. With g increasing the coefficients a and b grow gradually aspiring to their asymptotical values $\gamma\delta$ and $\gamma\delta_3$ respectively.

One more detail of the interrelation of the two components of the current. Let us observe the motion of the ends (tips) of the scissors. Their displacements are determined essentially by the value of $\xi_x = (b + a)z$. In the region $0 < g < g_1$ coefficients a and b have the opposite signs. This means that two components of current move out of phase – the shear (irrotational) component bends the bodies of the scissors trying to resist to their rotation. This process is described very well and illustrated by the figures 1, 2 in the paper by R. Hilton [3]. In the

region $g_1 < g < g_2$ coefficients a and b have the same sign. Therefore, both components move in phase – now the shear component bends the bodies of the scissors reinforcing their rotation. In the region $g_2 < g < \infty$ coefficients a and b again have opposite signs and the situation returns to the case of the first region. However, now the shear component dominates, hence it is better to say that the rotation tries to resist to the shear.

6 Concluding remarks

The low energy magnetic dipole strength produced in all QRPA calculations (even in the schematic model HO+QQ) is always distributed over several states in the region $0 < E < 4$ MeV, because each $\alpha\beta$ pair contributing to $B(M1)_{sc}$ occurs at a different energy $E_{\alpha\beta} = E_\alpha + E_\beta$ (see Introduction). The same picture is observed experimentally. These facts were in sharp contradiction with the anticipated single peak, predicted in early papers [14, 12, 15] and caused “controversy about the collectivity and the correct interpretation - in the sense of classical motion - of the orbital magnetic dipole modes [28]”. It became clear that one needs some kind of a bridge between the classical and quantum mechanical approaches. For the description of this phenomenon “a formalism is required in which the interplay between the collective and single particle aspects of the system are adequately treated [29]”.

Some authors “have tried to complement RPA calculations with realistic forces by classical and semiclassical methods which are flexible enough also to admit other solutions than the one anticipated [28]”. This way however does not give a satisfactory solution of the problem, because usually the phenomenological models have not the direct connection with quantum mechanical ones. The most popular way to extract the underlying physical nature of the state from microscopic (RPA or QRPA) calculations is the calculation of overlaps [2]. However, the overlaps usually do not give the full information about the various properties of the studied phenomenon. As a matter of fact, this procedure answers only the question: “What part of the mode strength is excited by this operator (with which the overlap is calculated)?” Usually one calculates the overlap with the “synthetic scissors” [4, 25]. It is known, however, that due to the coupling with the IVGQR [12], the low-energy magnetic dipole mode contains rather big isovector plus isoscalar admixtures of two orthogonal shears, as shown in [29] by calculating

the overlap with the proper operator. Nevertheless, this overlap does not exhaust all strength. What other operators are necessary to clarify completely the nature of the considered mode?

We think that the combination of two complementary methods, namely RPA (or QRPA) and WFM method, can be very useful. Based on the same approach (time-dependent HF or HFB together with a small amplitude approximation), they allow one, starting from the same Hamiltonian with the same forces, to obtain either (by RPA calculations) the refined microscopic structure or (applying WFM method) the crude “macroscopic” picture (which reminds very much the results of semiclassical approaches) of the same phenomenon. This interrelation of RPA and WFM was investigated in [30]. In particular, the identity of both methods in the case of a schematic model was demonstrated there. With the help of the WFM method, taking into account moments of higher and higher rank, one can produce a more and more detailed description of the phenomenon, achieving (at least in principle) the maximally fragmented picture given by the experiment (and RPA).

One more remark is in order. Discussing the scissors mode energy and its δ -dependence, one has usually in mind the mean excitation energy (centroid) defined as the center of gravity of the M1 strength distributed among the low-lying $K^\pi = 1^+$ states

$$E_{sc} = \sum_i E_i B(M1)_i / \sum_i B(M1)_i.$$

The sums are evaluated in the energy intervals around 3 MeV. The evaluation interval is not strictly determined. For example, Hamamoto and Nazarewicz [25] took $0 < E < 10$ MeV in their calculations of superdeformed nuclei. Applying the WFM method, which produces centroids of resonances, we avoid such problems.

In conclusion, the WFM method is generalized to take into account pair correlations and is used to calculate energies and transitions probabilities of the scissors mode. Excellent qualitative and reasonable quantitative agreement with experimental data is obtained. In addition the interrelation of microscopic and semiclassical features of the scissors mode is clarified.

Acknowledgements

Fruitful discussions with A. A. Dzhioev are gratefully acknowledged.

Appendix A

We remind that each ingredient of Eqs. (8) is a matrix in coordinates and spin projections. For example, $\hat{\kappa} \equiv \hat{\kappa}_{\mathbf{r},\sigma;\mathbf{r}',\sigma'}$. The spin structures of the normal and abnormal density are explicitly

$$\hat{\rho}_{\mathbf{r},\sigma;\mathbf{r}',\sigma'} = \begin{pmatrix} \hat{\rho}_{\mathbf{r},\mathbf{r}'} & 0 \\ 0 & \hat{\rho}_{\mathbf{r},\mathbf{r}'} \end{pmatrix}, \quad \hat{\kappa}_{\mathbf{r},\sigma;\mathbf{r}',\sigma'} = \begin{pmatrix} 0 & \hat{\kappa}_{\mathbf{r},\mathbf{r}'} \\ -\hat{\kappa}_{\mathbf{r},\mathbf{r}'} & 0 \end{pmatrix},$$

or

$$\hat{\rho}_{\mathbf{r},\uparrow;\mathbf{r}',\uparrow} = \hat{\rho}_{\mathbf{r},\downarrow;\mathbf{r}',\downarrow}, \quad \hat{\kappa}_{\mathbf{r},\uparrow;\mathbf{r}',\downarrow} = -\hat{\kappa}_{\mathbf{r},\downarrow;\mathbf{r}',\uparrow}.$$

With the help of the standard expression for the product of two matrices

$$(AB)_{\mathbf{r},\sigma;\mathbf{r}',\sigma'} = \int d^3r'' \sum_{\sigma''} A_{\mathbf{r},\sigma;\mathbf{r}'',\sigma''} B_{\mathbf{r}'',\sigma'';\mathbf{r}',\sigma'} = \int d^3r'' [A_{\mathbf{r},\sigma;\mathbf{r}'',\uparrow} B_{\mathbf{r}'',\uparrow;\mathbf{r}',\sigma'} + A_{\mathbf{r},\sigma;\mathbf{r}'',\downarrow} B_{\mathbf{r}'',\downarrow;\mathbf{r}',\sigma'}]$$

we find

$$\begin{aligned} (\hat{h}\hat{\rho})_{\mathbf{r},\uparrow;\mathbf{r}',\uparrow} &= \int d^3r'' \hat{h}_{\mathbf{r},\uparrow;\mathbf{r}'',\uparrow} \hat{\rho}_{\mathbf{r}'',\uparrow;\mathbf{r}',\uparrow}, \quad (\hat{h}\hat{\rho})_{\mathbf{r},\uparrow;\mathbf{r}',\downarrow} = (\hat{h}\hat{\rho})_{\mathbf{r},\downarrow;\mathbf{r}',\uparrow} = 0, \\ (\hat{h}\hat{\rho})_{\mathbf{r},\downarrow;\mathbf{r}',\downarrow} &= \int d^3r'' \hat{h}_{\mathbf{r},\downarrow;\mathbf{r}'',\downarrow} \hat{\rho}_{\mathbf{r}'',\downarrow;\mathbf{r}',\downarrow}, \\ (\hat{\Delta}\hat{\kappa})_{\mathbf{r},\uparrow;\mathbf{r}',\uparrow} &= \int d^3r'' \hat{\Delta}_{\mathbf{r},\uparrow;\mathbf{r}'',\downarrow} \hat{\kappa}_{\mathbf{r}'',\downarrow;\mathbf{r}',\uparrow}, \quad (\hat{\Delta}\hat{\kappa})_{\mathbf{r},\uparrow;\mathbf{r}',\downarrow} = (\hat{\Delta}\hat{\kappa})_{\mathbf{r},\downarrow;\mathbf{r}',\uparrow} = 0, \\ (\hat{\Delta}\hat{\kappa})_{\mathbf{r},\downarrow;\mathbf{r}',\downarrow} &= \int d^3r'' \hat{\Delta}_{\mathbf{r},\downarrow;\mathbf{r}'',\uparrow} \hat{\kappa}_{\mathbf{r}'',\uparrow;\mathbf{r}',\downarrow}. \end{aligned}$$

As an example we write out the first equation of (8) for $\sigma = \uparrow$ in detail:

$$\begin{aligned} i\hbar \dot{\hat{\rho}}_{\mathbf{r},\uparrow;\mathbf{r}',\uparrow} &= \int d^3r'' \left[\hat{h}_{\mathbf{r},\uparrow;\mathbf{r}'',\uparrow} \hat{\rho}_{\mathbf{r}'',\uparrow;\mathbf{r}',\uparrow} - \hat{\rho}_{\mathbf{r},\uparrow;\mathbf{r}'',\uparrow} \hat{h}_{\mathbf{r}'',\uparrow;\mathbf{r}',\uparrow} \right. \\ &\quad \left. - \hat{\Delta}_{\mathbf{r},\uparrow;\mathbf{r}'',\downarrow} \hat{\kappa}_{\mathbf{r}'',\downarrow;\mathbf{r}',\uparrow}^\dagger + \hat{\kappa}_{\mathbf{r},\uparrow;\mathbf{r}'',\downarrow} \hat{\Delta}_{\mathbf{r}'',\downarrow;\mathbf{r}',\uparrow}^\dagger \right]. \end{aligned} \quad (38)$$

The Wigner Transform (WT) of the single particle operator matrix $\hat{F}_{\mathbf{r}_1,\sigma;\mathbf{r}_2,\sigma'}$ is defined as

$$[\hat{F}_{\mathbf{r}_1,\sigma;\mathbf{r}_2,\sigma'}]_{WT} \equiv F_{\sigma,\sigma'}(\mathbf{r}, \mathbf{p}) = \int d^3s e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar} \hat{F}_{\mathbf{r}+\mathbf{s}/2,\sigma;\mathbf{r}-\mathbf{s}/2,\sigma'}$$

with $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$. It is easy to derive a pair of useful relations. The first one is

$$\begin{aligned} F_{\sigma,\sigma'}^*(\mathbf{r}, \mathbf{p}) &= \int d^3s e^{i\mathbf{p}\cdot\mathbf{s}/\hbar} \hat{F}_{\mathbf{r}+\mathbf{s}/2,\sigma;\mathbf{r}-\mathbf{s}/2,\sigma'}^* = \int d^3s e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar} \hat{F}_{\mathbf{r}-\mathbf{s}/2,\sigma;\mathbf{r}+\mathbf{s}/2,\sigma'}^* \\ &= \int d^3s e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar} \hat{F}_{\mathbf{r}+\mathbf{s}/2,\sigma';\mathbf{r}-\mathbf{s}/2,\sigma}^\dagger = [\hat{F}_{\mathbf{r}_1,\sigma';\mathbf{r}_2,\sigma}]_{WT} \end{aligned}$$

i.e. $[\hat{F}_{\mathbf{r}_1, \sigma; \mathbf{r}_2, \sigma'}^\dagger]_{WT} = [\hat{F}_{\mathbf{r}_1, \sigma'; \mathbf{r}_2, \sigma}]_{WT}^* = F_{\sigma' \sigma}^*(\mathbf{r}, \mathbf{p})$. The second relation is

$$\begin{aligned}\bar{F}_{\sigma \sigma'}(\mathbf{r}, \mathbf{p}) &\equiv F_{\sigma \sigma'}(\mathbf{r}, -\mathbf{p}) = \int d^3 s e^{i\mathbf{p} \cdot \mathbf{s} / \hbar} \hat{F}_{\mathbf{r} + \mathbf{s}/2, \sigma; \mathbf{r} - \mathbf{s}/2, \sigma'} \\ &= \int d^3 s e^{-i\mathbf{p} \cdot \mathbf{s} / \hbar} \hat{F}_{\mathbf{r} - \frac{\mathbf{s}}{2}, \sigma; \mathbf{r} + \frac{\mathbf{s}}{2}, \sigma'} = \int d^3 s e^{-i\mathbf{p} \cdot \mathbf{s} / \hbar} [\hat{F}_{\mathbf{r} + \mathbf{s}/2, \sigma'; \mathbf{r} - \mathbf{s}/2, \sigma}^\dagger]^*.\end{aligned}$$

For the hermitian operators $\hat{\rho}$ and \hat{h} this latter relation gives $[\hat{\rho}_{\mathbf{r}_1, \sigma; \mathbf{r}_2, \sigma}^*]_{WT} = \rho_{\sigma \sigma}(\mathbf{r}, -\mathbf{p})$ and $[\hat{h}_{\mathbf{r}_1, \sigma; \mathbf{r}_2, \sigma}^*]_{WT} = h_{\sigma \sigma}(\mathbf{r}, -\mathbf{p})$.

The Wigner transform of the product of two matrices F and G is

$$[\hat{F}\hat{G}]_{WT} = F(\mathbf{r}, \mathbf{p}) \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\Lambda}\right) G(\mathbf{r}, \mathbf{p}), \quad (39)$$

where the symbol $\overleftrightarrow{\Lambda}$ stands for the Poisson bracket operator

$$\overleftrightarrow{\Lambda} = \sum_{i=1}^3 \left(\frac{\overleftarrow{\partial}}{\partial r_i} \frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i} \frac{\overrightarrow{\partial}}{\partial r_i} \right).$$

For example the Wigner transform of Eq. (38) up to linear order in \hbar is

$$\begin{aligned}i\hbar \dot{f}_{\uparrow, \uparrow}(\mathbf{r}, \mathbf{p}) &= i\hbar \{h_{\uparrow, \uparrow}(\mathbf{r}, \mathbf{p}), f_{\uparrow, \uparrow}(\mathbf{r}, \mathbf{p})\} \\ &\quad - \Delta_{\uparrow, \downarrow}(\mathbf{r}, \mathbf{p}) \kappa_{\uparrow, \downarrow}^*(\mathbf{r}, \mathbf{p}) - \frac{i\hbar}{2} \{\Delta_{\uparrow, \downarrow}(\mathbf{r}, \mathbf{p}), \kappa_{\uparrow, \downarrow}^*(\mathbf{r}, \mathbf{p})\} \\ &\quad + \kappa_{\uparrow, \downarrow}(\mathbf{r}, \mathbf{p}) \Delta_{\uparrow, \downarrow}^*(\mathbf{r}, \mathbf{p}) + \frac{i\hbar}{2} \{\kappa_{\uparrow, \downarrow}(\mathbf{r}, \mathbf{p}), \Delta_{\uparrow, \downarrow}^*(\mathbf{r}, \mathbf{p})\},\end{aligned} \quad (40)$$

where $\{f, g\} \equiv f \overleftrightarrow{\Lambda} g = \sum_{i=1}^3 \left(\frac{\partial f}{\partial r_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial r_i} \right)$ is the Poisson bracket of arbitrary functions $f(\mathbf{r}, \mathbf{p})$ and $g(\mathbf{r}, \mathbf{p})$; $h(\mathbf{r}, \mathbf{p})$, $f(\mathbf{r}, \mathbf{p})$, $\Delta(\mathbf{r}, \mathbf{p})$ and $\kappa(\mathbf{r}, \mathbf{p})$ are Wigner transforms of $h_{\mathbf{r}_1, \mathbf{r}_2}$, $\rho_{\mathbf{r}_1, \mathbf{r}_2}$, $\Delta_{\mathbf{r}_1, \mathbf{r}_2}$ and $\kappa_{\mathbf{r}_1, \mathbf{r}_2}$ respectively. The functions h and f are real, because the matrices \hat{h} and $\hat{\rho}$ are hermitian. This example demonstrates in an obvious way that the dynamical equations (8) for the matrix elements $\rho_{\mathbf{r}, \sigma; \mathbf{r}', \sigma}$, $\rho_{\mathbf{r}, \sigma; \mathbf{r}', \sigma}^*$, $\kappa_{\mathbf{r}, \pm \sigma; \mathbf{r}', \mp \sigma}$, $\kappa_{\mathbf{r}, \pm \sigma; \mathbf{r}', \mp \sigma}^\dagger$, with $\sigma = \uparrow$ and $\sigma = \downarrow$ are transformed into eight dynamical equations for their Wigner transforms: 4 equations for $f_{\sigma, \sigma}(\mathbf{r}, \mathbf{p})$, $\bar{f}_{\sigma, \sigma}(\mathbf{r}, \mathbf{p})$, $\kappa_{\sigma, -\sigma}(\mathbf{r}, \mathbf{p})$, $\kappa_{\sigma, -\sigma}^*(\mathbf{r}, \mathbf{p})$ with $\sigma = \uparrow$ and 4 equations with $\sigma = \downarrow$. By definition $\bar{f}_{\sigma, \sigma}(\mathbf{r}, \mathbf{p}) = f_{\sigma, \sigma}(\mathbf{r}, -\mathbf{p})$. In the absence of spin dependent forces both of these subsets coincide and we can consider any one of them.

Appendix B

Let us consider in detail the arbitrary term of equations (11) proportional to \hbar^n . Such terms appear after expanding the exponent in formula (39). After integration with an arbitrary function $A(\mathbf{r}, \mathbf{p})$ we have

$$\begin{aligned} I_n &= \int d(\mathbf{p}, \mathbf{r}) A[f \overset{\leftarrow}{\Lambda}^n g] = \int d(\mathbf{p}, \mathbf{r}) \left(A \left[\frac{\partial f}{\partial x_i} \overset{\leftarrow}{\Lambda}^{n-1} \frac{\partial g}{\partial p_i} \right] - A \left[\frac{\partial f}{\partial p_i} \overset{\leftarrow}{\Lambda}^{n-1} \frac{\partial g}{\partial x_i} \right] \right) \\ &= - \int d(\mathbf{p}, \mathbf{r}) \left(\frac{\partial A}{\partial p_i} \left[\frac{\partial f}{\partial x_i} \overset{\leftarrow}{\Lambda}^{n-1} g \right] - \frac{\partial A}{\partial x_i} \left[\frac{\partial f}{\partial p_i} \overset{\leftarrow}{\Lambda}^{n-1} g \right] \right) \equiv A_p + A_x. \end{aligned} \quad (41)$$

Repeating the integration by parts we find

$$\begin{aligned} A_p &= - \int d(\mathbf{p}, \mathbf{r}) \frac{\partial A}{\partial p_i} \left(\left[\frac{\partial^2 f}{\partial x_i \partial x_j} \overset{\leftarrow}{\Lambda}^{n-2} \frac{\partial g}{\partial p_j} \right] - \left[\frac{\partial^2 f}{\partial x_i \partial p_j} \overset{\leftarrow}{\Lambda}^{n-2} \frac{\partial g}{\partial x_j} \right] \right) \\ &= \int d(\mathbf{p}, \mathbf{r}) \left(\frac{\partial^2 A}{\partial p_i \partial p_j} \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \overset{\leftarrow}{\Lambda}^{n-2} g \right] - \frac{\partial^2 A}{\partial p_i \partial x_j} \left[\frac{\partial^2 f}{\partial x_i \partial p_j} \overset{\leftarrow}{\Lambda}^{n-2} g \right] \right) \equiv A_{pp} + A_{px}. \\ A_x &= - \int d(\mathbf{p}, \mathbf{r}) \frac{\partial A}{\partial x_i} \left(\left[\frac{\partial^2 f}{\partial p_i \partial x_j} \overset{\leftarrow}{\Lambda}^{n-2} \frac{\partial g}{\partial p_j} \right] - \left[\frac{\partial^2 f}{\partial p_i \partial p_j} \overset{\leftarrow}{\Lambda}^{n-2} \frac{\partial g}{\partial x_j} \right] \right) \\ &= \int d(\mathbf{p}, \mathbf{r}) \left(\frac{\partial^2 A}{\partial x_i \partial p_j} \left[\frac{\partial^2 f}{\partial p_i \partial x_j} \overset{\leftarrow}{\Lambda}^{n-2} g \right] - \frac{\partial^2 A}{\partial x_i \partial x_j} \left[\frac{\partial^2 f}{\partial p_i \partial p_j} \overset{\leftarrow}{\Lambda}^{n-2} g \right] \right) \equiv A_{xp} + A_{xx}. \end{aligned} \quad (42)$$

Repeating once again the integration by parts we get

$$\begin{aligned} A_{pp} &= \int d(\mathbf{p}, \mathbf{r}) \frac{\partial^2 A}{\partial p_i \partial p_j} \left(\left[\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \overset{\leftarrow}{\Lambda}^{n-3} \frac{\partial g}{\partial p_k} \right] - \left[\frac{\partial^3 f}{\partial x_i \partial x_j \partial p_k} \overset{\leftarrow}{\Lambda}^{n-3} \frac{\partial g}{\partial x_k} \right] \right) \\ &= - \int d(\mathbf{p}, \mathbf{r}) \left(\frac{\partial^3 A}{\partial p_i \partial p_j \partial p_k} \left[\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \overset{\leftarrow}{\Lambda}^{n-3} g \right] - \frac{\partial^3 A}{\partial p_i \partial p_j \partial x_k} \left[\frac{\partial^3 f}{\partial x_i \partial x_j \partial p_k} \overset{\leftarrow}{\Lambda}^{n-3} g \right] \right). \end{aligned} \quad (43)$$

It is easy to see from the structure of A_{pp} (and A_{px} , A_{xp} , A_{xx}) that in the case, when $A(\mathbf{r}, \mathbf{p})$ is a polynomial of an order k , all integrals I_n with $n > k$ are equal to zero. In our case $k = 2$ and we find from the above formulae, that $I_1 = \int d(\mathbf{p}, \mathbf{r}) A\{f, g\}$, $I_2 = \int d(\mathbf{p}, \mathbf{r}) A\{\{f, g\}\}$, where $\{f, g\}$ is defined in Appendix A and

$$\{\{f, g\}\} \equiv f(\mathbf{r}, \mathbf{p}) \overset{\leftarrow}{\Lambda}^2 g(\mathbf{r}, \mathbf{p}) = \sum_{i,j=1}^3 \left(\frac{\partial^2 f}{\partial r_i \partial r_j} \frac{\partial^2 g}{\partial p_i \partial p_j} - 2 \frac{\partial^2 f}{\partial r_i \partial p_j} \frac{\partial^2 g}{\partial p_i \partial r_j} + \frac{\partial^2 f}{\partial p_i \partial p_j} \frac{\partial^2 g}{\partial r_i \partial r_j} \right).$$

Appendix C: Thomas-Fermi approach to nuclear pairing

Following Ref.[31], we define the density matrix averaged on the energy shell as

$$\hat{\rho}_E = \frac{1}{\tilde{g}(E)} \tilde{\delta}(E - \hat{H}) = \frac{1}{\tilde{g}(E)} \sum_{\nu} \tilde{\delta}(E - \varepsilon_{\nu}) |\nu\rangle \langle \nu|. \quad (44)$$

which is a smooth function of E since $\tilde{\delta}$ denotes a smeared delta function. The smeared level density $\tilde{g}(E)$ (per spin and isospin in this paper) in the denominator of expression (44) ensures the right normalization of $\hat{\rho}_E$. The smooth quantities entering in (44) are evaluated by replacing \hat{H} , the independent-particle Hamiltonian, by its classical counterpart H_{cl} which corresponds to the Thomas-Fermi (TF) approximation [20, 32]. This approach is not limited to the evaluation of expectation values of single particle operators. Also the average behavior of two-body matrix elements can be calculated [31]. In this paper we are interested in the semiclassical evaluation of the average pairing matrix elements which at TF level read

$$v(E, E') = \frac{1}{\tilde{g}(E)\tilde{g}(E')} \sum_{\nu, \nu'} \tilde{\delta}(E - \varepsilon_{\nu}) \tilde{\delta}(E' - \varepsilon_{\nu'}) \langle \Phi(\nu, \bar{\nu}) | v | \Phi(\nu', \bar{\nu}') \rangle, \quad (45)$$

where $|\Phi(\nu, \bar{\nu})\rangle$ is an *antisymmetric* normalized two-body state constructed out of a state $|\nu\rangle$ and its time-reversed state $|\bar{\nu}\rangle$. As it is known [20, 32, 33], the Strutinsky method averages the density matrix over an energy interval corresponding roughly to the distance between two major shells. Implicitly the same holds if the equivalent Wigner-Kirkwood expansion (TF approximation at lowest order) is used for obtaining $\hat{\rho}_E$.

As far as we are interested in the semiclassical counterpart of the density matrix $\hat{\rho}_E$ on the energy shell, we start considering its Wigner transform $f_E(\mathbf{r}, \mathbf{p})$. In order to obtain the pure TF approximation, we differentiate with respect to E the Wigner-Kirkwood expansion of the full single-particle one-body density matrix $\hat{\rho} = \Theta(E - \hat{H})$ retaining only the leading term, which reads after normalization:

$$f_E^{TF}(\mathbf{r}, \mathbf{p}) = \frac{1}{\tilde{g}(E)} \delta(E - H_{cl}), \quad (46)$$

where $H_{cl} = p^2/2m^* + V(\mathbf{r})$ is the classical Hamiltonian of independent particles with a constant effective mass m^* moving in an external potential well. Integration over the momentum yields the local density on the energy shell:

$$\rho_E^{TF}(\mathbf{r}) = \frac{1}{(2\pi\hbar)^3} \int d^3p f_E^{TF}(\mathbf{r}, \mathbf{p}) = \frac{m^* k_E(\mathbf{r})}{2\pi^2 \hbar^2 \tilde{g}(E)}, \quad (47)$$

where the local momentum at the energy E is

$$k_E(\mathbf{r}) = \frac{p_E(\mathbf{r})}{\hbar} = \sqrt{\frac{2m^*}{\hbar^2}(E - V(\mathbf{r}))} \quad (48)$$

and the level density $\tilde{g}(E)$ is given by the integral of the local level density $\tilde{g}(E, \mathbf{r})$

$$\tilde{g}(E) = \int d^3r \tilde{g}(E, \mathbf{r}) = \int d^3r \frac{m^* k_E(\mathbf{r})}{2\pi^2 \hbar^2} \quad (49)$$

Now we proceed to calculate the average pairing matrix elements $v(E, E')$ of the Gogny D1S force [34] which is known to reproduce the experimental gap values when used in microscopic Hartree-Fock-Bogolyubov calculations [35]. Starting from (45) and following the method explained in detail in Ref. [31], one arrives in TF approximation at

$$v(E, E') = \int d^3r \int \frac{d^3p d^3p'}{(2\pi\hbar)^6} f_E^{TF}(\mathbf{r}, \mathbf{p}) v(\mathbf{p} - \mathbf{p}') f_{E'}^{TF}(\mathbf{r}, \mathbf{p}') \quad (50)$$

where f_E^{TF} is given by equation (46) and $v(\mathbf{p} - \mathbf{p}')$ is the Fourier transform of the particle-particle part of the Gogny force which describes the pairing. As far as the only dependence on \mathbf{p} and \mathbf{p}' is in $v(\mathbf{p} - \mathbf{p}')$, we can average over the angle between \mathbf{p} and \mathbf{p}' as follows:

$$v(p, p') = \frac{1}{4\pi} \int v(\mathbf{p} - \mathbf{p}') d\Omega. \quad (51)$$

This result and the fact that the TF on shell density (46) can be recast as $f_E^{TF}(\mathbf{r}, \mathbf{p}) = m^* \delta(p - p_E) / (\tilde{g}(E) p_E)$ allows one to perform easily the angular integral in (50), with the result

$$v(E, E') = \frac{1}{\tilde{g}(E) \tilde{g}(E')} \frac{1}{4\pi^3} \left(\frac{2m^*}{\hbar^2} \right)^2 \int dr r^2 k_E k_{E'} v(p, p') \quad (52)$$

which in the particular case of the Gogny force reads:

$$v(E, E') = \sum_{i=1}^2 \frac{z_i}{\mu_i^2} \frac{1}{2\pi^3 \tilde{g}(E) \tilde{g}(E')} \left(\frac{2m^*}{\hbar^2} \right)^2 \int_0^{R_t} dr r^2 \exp \left\{ -\frac{\mu_i(k_E^2 + k_{E'}^2)}{4} \right\} \sinh \frac{\mu_i^2 k_E k_{E'}}{2}, \quad (53)$$

where R_t is the classical turning point and $z_i = \pi^{3/2} \mu_i^3 (W_i - B_i - H_i + M_i)$. The factors z_i correspond to pairing in the $S = 0$ and $T = 1$ channel and are written in terms of the parameters of the Gogny force W_i , B_i , H_i , M_i and μ_i [34].

The semiclassical TF gap equation reads

$$\tilde{\Delta}(E) = - \int_{V_1}^{V_2} dE' \tilde{v}(E, E') \tilde{g}(E') \frac{\tilde{\Delta}(E')}{2\sqrt{(E' - \mu)^2 + \tilde{\Delta}(E')^2}} \quad (54)$$

where V_1 and V_2 are the lower and upper limits of the pairing window and μ is the chemical potential which is obtained by the condition of the neutron (proton) number, i.e. integrating the corresponding TF level density up to the Fermi level:

$$N_\tau = 2 \int_{V_0}^{E_{F_\tau}} \tilde{g}(E) dE, \quad (55)$$

where V_0 is the bottom of the potential $V(\mathbf{R})$ and E_{F_τ} is the Fermi level for each type of particles ($\tau = n, p$).

The pairing density in the TF approximation is given by:

$$\tilde{\kappa}(E) = \frac{\tilde{\Delta}(E)}{2\sqrt{(E - \mu)^2 + \tilde{\Delta}(E)^2}} \quad (56)$$

Next we obtain the gap and the pairing density in coordinate space by integrating over E the gap and pairing density given by Eqs. (54) and (56), respectively, weighted with the local level density $\tilde{g}(E, \mathbf{R})$:

$$\tilde{\Delta}(\mathbf{r}) = \int dE \tilde{g}(E, \mathbf{r}) \tilde{\Delta}(E) \quad (57)$$

and

$$\tilde{\kappa}(\mathbf{r}) = \int dE \tilde{g}(E, \mathbf{r}) \tilde{\kappa}(E), \quad (58)$$

In the calculation of the energy and $B(M1)$ factors of the scissors mode the zeroth and fourth order moments of the pairing density for each kind of nucleons are needed, they read:

$$k_0 = \int d^3r \tilde{\kappa}(\mathbf{r}) = 4 \int dE \tilde{\kappa}(E) \int d^3r \tilde{g}(E, \mathbf{r}) \quad (59)$$

and

$$k_4 = \int d^3r \tilde{\kappa}(\mathbf{r}) x^2 z^2 = 4 \int dE \tilde{\kappa}(E) \int d^3r x^2 z^2 \tilde{g}(E, \mathbf{r}), \quad (60)$$

where the factor four takes into account the spin-isospin degeneracy. In order to calculate Eqs. (59) and (60), we use a single particle potential of harmonic oscillator type. The calculation of $\tilde{\kappa}(E)$ (Eq. (56)) is carried out in spherical symmetry and the deformation is included in $\tilde{g}(E, \mathbf{R})$ in order to obtain the moments of the pairing density. With the harmonic oscillator potential the integral in coordinate space can be done analytically and the calculation of the k_0 and k_4 moments reduce to the following integrals over E :

$$k_0 = \frac{1}{2\hbar^3 \omega_x^2 \omega_z} \int dE E^2 \kappa(E) \quad (61)$$

and

$$k_4 = \frac{1}{6} \left(\frac{\hbar^2}{2m^*} \right)^2 \frac{1}{\hbar^7 \omega_x^4 \omega_z^3} \int dE E^4 \tilde{\kappa}(E), \quad (62)$$

where ω_x^2 and ω_z^2 are given by formula (22) and the deformation δ is taken from the experiment [17]. In all the calculation an effective mass $m^* = 0.8m$ has been used.

Appendix D: Pair correlations in the superfluid model of deformed atomic nuclei

The microscopical calculations of integrals k_0^τ and k_4^τ (19) are performed with the single particle wave functions of the deformed nuclei. These functions are obtained by the numerical solution of the Schrödinger equation with the axially deformed Woods–Saxon potential including the spin–orbit interaction:

$$V_{WS}(\mathbf{r}) = -V_0 / \{1 + \exp[\alpha(r - R(\theta))]\},$$

$$R(\theta) = R_0[1 + \beta_0 + \beta_2 Y_{20}(\theta) + \beta_4 Y_{40}(\theta)],$$

$$V_{ls}(\mathbf{r}) = -\kappa(\mathbf{p} \times \sigma) \nabla V_{WS}.$$

Here $R_0 = r_0 A^{1/3}$; the constant β_0 is introduced to ensure the volume conservation; β_2 and β_4 are quadrupole and hexadecapole deformation parameters. The details of the method of the solution can be found in [36, 37, 38, 39].

The wave function of the deformed nucleus is represented in a form of a superposition

$$|\nu\rangle \equiv \Psi_\Omega^\rho = \sum_{nlj} a_{nlj}^{\Omega\rho} \cdot \Psi_{nlj}^\Omega, \quad (63)$$

where $a_{nlj}^{\Omega\rho}$ are the expansion coefficients and

$$\Psi_{nlj}^\Omega = R_{nlj}(r) \mathbf{Y}_{lj}^\Omega \quad (64)$$

are the wave functions of the spherical basis. Here \mathbf{Y}_{lj}^Ω are the spherical spinors and $R_{nlj}(r)$ are eigenfunctions of the radial part of the Schrödinger equation with the spherical Woods–Saxon potential. This paper deals with the single particle matrix elements of the type $\langle \nu | F(r) Y_{\lambda 0} | \nu \rangle$, with $F(r)$ and $Y_{\lambda 0}$ being the radial part and the spherical function respectively.

Table 2: β_2 , β_4 , V_0 (in MeV), r_0 (in fm) and α (in fm^{-1}) are parameters of the deformed Woods–Saxon potential; G (in MeV) are pairing strength constants; κ are spin–orbit parameters in fm^2 .

A	β_2	β_4	Neutron system					Proton system				
			V_0	r_0	α	κ	G	V_0	r_0	α	κ	G
135	0.14	0	47.0	1.26	1.67	0.40	0.141	58.0	1.24	1.578	0.35	0.150
147	0.13	0	46.8	1.26	1.67	0.40	0.123	57.4	1.24	1.578	0.35	0.138
155	0.30	0.04	47.2	1.26	1.67	0.40	0.115	59.2	1.24	1.69	0.36	0.154
165	0.28	0.02	44.8	1.26	1.67	0.43	0.110	59.2	1.25	1.63	0.355	0.132
173	0.26	-0.02	44.8	1.26	1.67	0.42	0.108	59.2	1.25	1.59	0.32	0.133
181	0.20	-0.03	43.4	1.26	1.67	0.40	0.106	59.8	1.24	1.67	0.33	0.130
193	0.14	0	43.4	1.26	1.67	0.40	0.101	59.8	1.24	1.67	0.33	0.121

Pair correlations are taken into account in the frame of the BCS theory. The interaction leading to the superfluid pairing correlations acts between the particles in time–reversed conjugate states. The pairing matrix element $G(\nu+, \nu-; \nu'-, \nu'+)$ is usually assumed to be a constant G independent of ν and ν' [19]. In this approximation the gap Δ does not depend on ν either: $\Delta = G \sum_{\nu} u_{\nu} v_{\nu}$. Then the equations determining Δ and the chemical potential μ read:

$$\frac{2}{G} = \sum_{\nu} \frac{1}{\epsilon_{\nu}}, \quad (66)$$

$$N = 2 \sum_{\nu} v_{\nu}^2.$$

Here N is the number of particles, u_{ν} and v_{ν} are the coefficients of the Bogoliubov canonical transformation, $\epsilon_{\nu} = (\Delta^2 + (E_{\nu} - \mu)^2)^{1/2}$ is the quasiparticle energy, E_{ν} is the energy of the single particle state (63). After all transformations the integrals k_0^{τ} and k_4^{τ} can be written as

$$k_0^{\tau} = 4 \int d^3r \kappa_+^{\tau 0}(\mathbf{r}) = 4 \sum_{\nu} u_{\nu} v_{\nu} \sum_{nlj} (a_{nlj}^{\nu})^2, \quad (67)$$

$$\begin{aligned} k_4^{\tau} &= 4 \int d^3r x^2 z^2 \kappa_+^{\tau 0}(\mathbf{r}) = \\ &= \frac{8\sqrt{\pi}}{15} \sum_{\nu} u_{\nu} v_{\nu} \langle \nu | r^4 (Y_{00} + \frac{\sqrt{5}}{7} Y_{20} - \frac{4}{7} Y_{40}) \kappa_+^{\tau 0}(\mathbf{r}) | \nu \rangle = \\ &= \sum_{\Omega, \rho} u_{\Omega \rho} v_{\Omega \rho} \sum_{nljn'l'j'} a_{nlj}^{\Omega \rho} a_{n'l'j'}^{\Omega \rho} I_{n'l'j'}^{nlj} \sum_{\lambda=0,2,4} L_{\lambda} C_{j\Omega, \lambda 0}^{j'\Omega} C_{j'1/2, \lambda 0}^{j1/2}, \end{aligned} \quad (68)$$

where $I_{n'l'j'}^{nlj} = \int r^6 R_{nlj}(r) R_{n'l'j'}(r) dr$, coefficients L_λ are: $L_0 = \frac{4}{15}$, $L_2 = \frac{4}{15}$, $L_4 = -\frac{16}{35}$, $C_{j_1 m_1, j_2 m_2}^{JM}$ is the Clebsh–Gordan coefficient.

All considered nuclei were divided into several groups. The deformation of the potential V_{WS} in each group was chosen close to the average value of experimental deformations of nuclei of the group. The schemes of the single particle states for each of these groups were calculated with the fixed set of parameters, each set being fitted to achieve a correct sequence of the single particle levels in deformed nuclei. The sets of these parameters for neutron and proton systems are given in Table 2. The parameters for $A=155, 165, 173, 181$ are taken from Ref. [40]. All discrete and quasidiscrete levels in the interval from the bottom of the potential well up to the energy 22 MeV were taken into account in the calculations. Such basis allows one to study low lying states and giant resonances as well. The pairing strength constants G were adjusted as a function of the size of the single particle basis.

References

- [1] E. B. Balbutsev and P. Schuck, Nucl. Phys. A **720** 293 (2003);
E. B. Balbutsev and P. Schuck, Nucl. Phys. A **728** 471 (2003).
- [2] D. Zawischa, J. Phys. G: Nucl. Part. Phys. **24** 683 (1998).
- [3] R. R. Hilton, Ann. Phys. (N.Y.) **214** 258(1992).
- [4] N. Lo Iudice, La Rivista del Nuovo Cimento **23** N9 (2000).
- [5] D. Bohle et al., Phys. Lett. B **137** 27 (1984).
- [6] W. Ziegler, C. Rangacharyulu, A. Richter, C. Spieler, Phys. Rev. Lett. **65** 2515 (1990).
- [7] H. H. Pitz et al., Nucl. Phys. A **509** 587 (1990).
- [8] J. Margraf et al., Phys. Rev. C **47** 1474 (1993).
- [9] C. Rangacharyulu et al., Phys. Rev. C **43** R949 (1991).
- [10] N. Pietralla et al., Phys. Rev. C **52** R2317 (1995).

- [11] J. Enders et al., Phys. Rev. C **59** R1851 (1999).
- [12] E. Lipparini, S. Stringari, Phys. Lett. B **130** 139 (1983).
- [13] E. Lipparini, S. Stringari, Phys. Rep. **175** 103 (1989).
- [14] T. Suzuki, D. J. Rowe, Nucl. Phys. A **289** 461 (1977).
- [15] D. R. Bes, R. A. Broglia, Phys. Lett. B **137** 141 (1984).
- [16] I. Hamamoto, C. Magnusson, Phys. Lett. B **260** 6 (1991).
- [17] N. Pietralla et al., Phys. Rev. C **58** 184(1998).
- [18] E. Garrido et al., Phys. Rev. C **44** R1250 (1991).
- [19] V. G. Soloviev, *Theory of Complex Nuclei* (Nauka, Moscow, 1971; Oxford, Pergamon Press, 1976).
- [20] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, Berlin, 1980).
- [21] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale University Press, New Haven, Conn., 1969).
- [22] E. B. Balbutsev, Sov. J. Part. Nucl. **22** 159 (1991).
- [23] E. B. Balbutsev and P. Schuck, Nucl. Phys. A **652** 221 (1999).
- [24] A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975), Vol. 2.
- [25] I. Hamamoto and W. Nazarewicz, Phys. Lett. B **297** 25 (1992).
- [26] M. Matsuo, Y. Serizawa and K. Mizuyama, arXiv:nucl-th/0608048 v1.
- [27] V. G. Soloviev, A. V. Sushkov, N. Yu. Shirikova and N. Lo Iudice, Nucl. Phys. A **600** 155 (1996),
V. G. Soloviev, A. V. Sushkov, N. Yu. Shirikova, Phys. Rev. C **53** 1022 (1996).
- [28] M. Macfarlane, J. Speth and D. Zawischa, Nucl. Phys. A **606** 41 (1996).

- [29] R. R. Hilton *et al.*, in *Proceedings of 1st International Spring Seminar on Nuclear Physics “Microscopic Approaches to Nuclear Structure”, Sorrento, 1986*, Ed. by A. Covello (Bologna, Physical Society, 1986).
- [30] E. B. Balbutsev and P. Schuck, *Physics of Atomic Nuclei* **69** 1985 (2006).
- [31] X. Viñas, P. Schuck, M. Farine and M. Centelles, *Phys. Rev. C* **67** 054307 (2003).
- [32] M. Brack and R. K. Badhuri, *Semiclassical Physics* (Addison-Wesley, Reading, MA, 1997).
- [33] M. Brack and H. C. Pauli, *Nucl. Phys. A* **207**, 401 (1973); B. K. Jennings, *Nucl. Phys. A* **207** 538 (1973).
- [34] J. F. Berger, M. Girod, and D. Gogny, *Comput. Phys. Commun.* **63** 365 (1991); *Nucl. Phys. A* **502** 85c (1989).
- [35] M. Kleban, B. Nerlo-Pomorska, J. F. Berger, J. Dechargé, M. Girod, and S. Hilaire, *Phys. Rev. C* **65** 024309 (2002).
- [36] F. A. Gareev, S. P. Ivanova, B. N. Kalinkin, *Izvestiya AN SSSR, ser. fiz.* **33** 1690 (1968).
- [37] F. A. Gareev, S. P. Ivanova, L. A. Malov and V.G. Soloviev, *Nucl. Phys. A* **171** 197 (1971).
- [38] F. A. Gareev, S. P. Ivanova, V. G. Soloviev, S. I. Fedotov, *Sov. J. Part. Nucl.* **4** 357 (1973).
- [39] L. A. Malov, V. O. Nesterenko, N. Yu. Shirikova, JINR report, R4-83-811 (Dubna, 1983).
- [40] L. A. Malov, V. G. Soloviev, *Sov. J. Part. Nucl.* **11** 301 (1980).